Modeling of Incident Waves near the Ship's Hull (Application of Autoregressive Approach in Problems of Simulation of Rough Seas)

Alexander B. Degtyarev
St. Petersburg State University, Faculty of Applied Mathematics and Control Processes

Arthur M. Reed
David Taylor Model Basin (NSWC/CD)

ABSTRACT

A holistic approach to the problem of modeling irregular waves for simulation of ship behavior is proposed. The mathematical basis considered is a representation of the stochastic processes of interest by a generalized autoregressive model (ARM) and related models of periodically correlated random process. ARM is used to generate space-time realizations of the wave surface of any duration. Application of the aforesaid model takes into account both the true stochastic process (lack of repetition in quasi-steady implementation of the process inherent in other models of wind waves), and synoptic and seasonal variability of wave forecasts for a specific geographic region can be incorporated.

KEYWORDS

Autoregressive modelling; Sea state modelling ; Synoptic and seasonal variability of waves

INTRODUCTION

Any research related to the study of the behavior of a ship at sea, requires a description of wind waves, which are the major disturbance that displaces the vessel from equilibrium. Currently, the most popular models for describing wind waves, are models based on the linear expansion of a stochastic moving surface as a system of independent random variables. These include models by St. Denis & Pearson (1953), Rosenblatt (1957), Sveshnikov (1959), and Longuet-Higgins (1962). The most popular model is that of Longuet-Higgins, which is based on a stochastic approximation of the moving wave-front as a superposition of elementary harmonic waves with random phases and random amplitudes, ε_n, c_n:

$$\varsigma(x,y,t) = \sum_n c_n \cos(u_n x + v_n y - \omega_n t + \varepsilon_n)\,, \qquad (1)$$

where the wave number (u_n, v_n) is continuously distributed on the (u, v) plane, i.e. the unit area $du \times dv$ contains an infinite number of wave numbers. The frequency ω_n associated with wave numbers (u_n, v_n) is given by a dispersion relation

$$\omega_n = \omega(u_n, v_n)\,.$$

The phase ε_n are jointly independent random variables uniformly distributed in the interval $[0, 2\pi]$.

Longuet-Higgins showed that under the above conditions, the function $\zeta(x, y, t)$ is a three-dimensional steady-state homogeneous ergodic Gaussian field, defined by

$$2E_\varsigma(u,v)dudv = \sum_n c_n^2\,.$$

where $E_\varsigma(u,v)$ is two dimensional spectral density of wave energy.

Longuet-Higgins' model is simple and is easily computed. It incorporates the physical fundamentals of the process of wind waves and is fully consistent with the task of modeling ocean waves.

Indeed, consider one of the simplest general conservation laws, the law of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{V}\right) = 0\,, \qquad (2)$$

where ρ is the density of the liquid, and V the fluid velocity.

In relation to ocean waves, we can make the assumptions of incompressibility and isotropy within the marine environment. In this case equation (2) reduces to

$$div\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = 0 \qquad (3)$$

The physics of wind waves is defined primarily by the action of gravitational forces, which simplifies nature of the phenomenon under investigation. This approach allows us to consider the irrotational motion of the fluid and introduce the velocity potential φ. Then equation (2) & (3) reduce to Laplace's equation, which is the most general field equation for the problem of wave motions of a liquid:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

The difference between one formulation or another of the wave problem will be in the nature of the boundary conditions satisfied at the surface of the air-water interface. In our case, it is important to remember that the linear formulation of the boundary conditions is (a) applied on the surface of the undisturbed fluid ($z = 0$) and (b) all nonlinear terms in the boundary conditions are ignored. The Laplace equation is linear and its solution can be found using Fourier transforms. Thus, for plane waves a well-known solution is given in the form of a definite integral (Kochin, et al., 1964).

$$\varphi(x,z,t) = \int_0^\infty e^{kz}\left[A(k,t)\cos kx + B(k,t)\sin kx\right]dk$$

A similar, but slightly more complicated solution is obtained for the three-dimensional case. The constants A and B are determined from the boundary conditions on the surface. In the linear formulation the equation of the wave profile is

$$\zeta(x,t) = -\frac{1}{g}\frac{\partial\varphi(x,0,t)}{\partial t}$$

$$= \int_0^\infty \left[\frac{\partial A(k,t)}{\partial t}\cos kx + \frac{\partial B(k,t)}{\partial t}\sin kx\right]dk$$

$$= \int_0^\infty C_t(k,t)\cos\left(kx + \varepsilon(k,t)\right)dk \qquad (4)$$

If we set $c_n = C_t(k_n, t)\,dk$, then wave model (1) may be associated with an approximation of integral (4).

Thus, Longuet-Higgins' model is distinguished by its considerable clarity and the simplicity of the computational algorithm. However, it is not without some serious shortcomings inherent in models of this class:

- The Longuet-Higgins model is only designed to represent a stationary Gaussian field. Normal distribution of the simulated process (1) is a consequence of the central limit theorem. Its application to the analysis of more general problems (e.g., the evolution of ocean waves in a storm, or the study of ocean waves distorted in shallow water) represents a significant challenge.
- Models of this class are periodic and need a very large set of frequencies to re-creating long-term simulation.
- In the numerical implementation of the Longuet-Higgins model, it appears that the rate of convergence of (1) is very slow. This is seen as a distortion of the energy spectrum of the simulated process (i.e. not provided by the convergence of (u, v)), and the laws for the distribution of elementary waves, especially in terms of extreme events (not ensured by the convergence of ε). This problem becomes especially significant when simulating complex waves that have a broad spectrum with many-peaks.

This latter point becomes particularly critical in numerical simulation. In a time domain computation of the responses of a vessel in a random seaway, the repeated evaluation of the apparently simple equation, (1) at hundreds of points on the hull for thousands of time steps can becomes a major factor determining the execution speed of the code (Beck & Reed, 2001). This becomes an even more significant issue in a nonlinear computation where the wave model is even more complex. Thus identifying a significantly less time intensive method for modelling the ambient ocean-wave environment has the potential for significantly speeding the total simulation process.

AUTOREGRESSIVE MODEL OF OCEAN WAVES

Another approach to modeling wind waves is possible in terms of the representation of a stochastic moving surface as a linear transforma-

tion of white noise with memory. These methods are one of the most popular ways of modeling stationary ergodic Gaussian random processes with given correlation characteristics (Box, et al., 2008). However, these methods have were not used to simulate wind waves for a long time. The first attempts to model two-dimensional disturbances were undertaken in the early 70's (cf. Kostecki, 1972), and the impetus for this was the development of the resonance theory of waves in wind. However, the formal mathematical framework was developed by Gurgenidze & Trapeznikov (1988) and Rozhkov & Trapeznikov (1990). They built a one-dimensional model of ocean waves $\zeta(t)$, on the basis of an autoregressive-moving average (ARMA) model

$$\zeta_t = \sum_{i=1}^{N} \Phi_i \zeta_t + \sum_{i=1}^{P} \Theta_j \varepsilon_{t-j} \qquad (5)$$

Here $\mathrm{F_j}$ are the autoregressive parameters, Θ_j the parameters of the moving average, and ε_j white noise with an infinitely divisible distribution law. Any ARMA process can be uniquely represented as a process moving average and autoregression process of general—infinite order (Gurgenidze & Trapeznikov, 1988), and the parameters of the spectral representation are defined by the rule of division of power series (in a rational factorized form, Rozhkov & Trapeznikov, 1990):

$$S(\omega) = \frac{\Delta\sigma^2}{\pi} \times \frac{\prod_m (1 - z_m \exp(-im\omega\Delta))(1 - z_m \exp(im\omega\Delta))}{\prod_n (1 - p_n \exp(-in\omega\Delta))(1 - p_n \exp(in\omega\Delta))}$$

where z_m and p_n are the zeros of the numerator (MA) and the denominator (AR), respectively, which form a pair of mutually conjugate numbers. If some of the zeros of the numerator are located close to the unit circle, then the spectral density will have pronounced dips. The use of the MA model is good in those cases where the approximated spectrum is unimodal with no distinct peaks but fails when the spectra has distinct peaks or is mutimodal. Additionally, the MA model has a rather low numerical stability. The AR model is well suited to describe spectra with pronounced peaks, which charactrizes the spectrum of ocean waves. In practice it has been traditional to use an autoregressive model, ie to represent the spectral density of time series in the form of a polynomial fraction (Box, *et al.*, 2008):

$$\zeta_t = \sum_{i=1}^{N} \Phi_i \zeta_{t-i} + \varepsilon_t;$$

$$S_\zeta(\omega) = \frac{\sigma_\varepsilon^2}{2\pi} \frac{\Delta}{\left| 1 + \sum_{j=1}^{N} \Phi_j \exp[-ij\Delta\omega] \right|^2} \qquad (6)$$

where ζ_t is the wave elevation at time t, N is the order of the model, Φ_i are coefficients to be computed using autocorrelation function (ACF), $K_\zeta(t)$, ζ_{t-i} are the N last realizations of ζ_t [i =1,...,N], and e_t is white noise with particular statistical characteristics. Here σ_ε^2 is the variance of white noise, and Δ the sampling interval of the series. Autoregressive coefficients can be estimated from the autocovariate function (ACF) using Yule-Walker equations. Details of the procedure for deriving AR coefficients estimates are shown in Appendix1.

Theoretically the number of autoregressive coefficients tends to infinity. In practice we have such an ACF that high order of autoregressive coefficients tend to zero, and we can neglect them.

To close the ARM (obtaining the ARM of a particular order) we can use several different approaches. For example, Box and Jenkins proposed to use least-squares method based on estimated values of ACF. However, Rozhkov & Trapeznikov (1990) have shown that in this case building of ARM is more favorable for on already smoothed ACF instead of on natural data.

A remarkable feature of ARMA models for wind waves is the small number of components (N), required to recreate the wave surface. In addition the stochastic nature and periodicity of such a model is determined solely by the period of random number generator, used in the simulation. The computational algorithm is relatively simple and converges fairly quickly on the spectral characteristics. Its development does not rely on the properties of probabilistic convergence to a Gaussian. On the basis of a linear transformation of white noise with memory, nonlinear transformation of white noise with memory to an arbitrary distribution law can easily be built into the model.

It is worth noting that the model of Longuet-Higgins is, in fact, a MA model (Rozhkov & Trapeznikov, 1990). However, MA and ARMA models are asymptotically equivalent, since each independent spectral component of formula (1) is a moving average process, which, in the addition of moving processes theorem is a moving average, leads to the satisfying the MA property for the Linguet-Higgins model (Gurgenidze & Trapeznikov, 1988).

More recently, the ARMA approach has been extended to any random scalar field *A* (Boukhanovsky & Degtyarev, 1995, 1996; Degtyarev & Boukhanovsky 2000; Boukhanovsky, *et al.* 2001). For example, a discrete model applicable to a moving wavy surface in three dimensions (2-D space + 1-D temporal) can be defined as

$$\varsigma_{(x,y,t)} = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\sum_{k=0}^{N_t}\Phi_{(i,j,k)}\varsigma_{(x-i,y-j,t-k)} + \varepsilon_{(x,y,t)} \quad (7)$$

where $\Phi_{(i,\,j,\,k)}$ is the generalized coefficients of the ARM, and ε(x, y, t) a field of white noise.

The procedure for estimating the parameters of the ARM and dispersion of the field of white noise, as well as the transition to the field with an arbitrary distribution is carried out in a manner similar to that of the one-dimensional version, although it has additional computational features (Boukhanovsky & Degtyarev, 1995, 1996; Degtyarev & Boukhanovsky, 2000).

In general, a discrete vector autoregressive process takes the form:

$$\zeta_{\vec{v}} = \sum_{\vec{j}=\vec{0}}^{\vec{N}}\Phi_{\vec{j}}\,\zeta_{\vec{v}-\vec{j}} + \sigma_\varepsilon^2\varepsilon_{\vec{v}}, \quad (8)$$

where the arrow over the corresponding index means a passage to multiple components of a scalar random process. As these components can be, for example, time + space, then we have three components (*x*, *y*, *t*). A component can act as any other scalar quantities such as temperature, salinity, concentration of any substance.

Now how we obtain Yule-Walker equations for scalar random fields. Let us multiply both sides of the multidimensional analog of (5) or (6) by $\varsigma(\vec{u})$ and average both sides (interval for averaging is whole dimension of the process variation). In this case we obtain the generalized Yule-Walker equations:

$$\begin{aligned} &K_\zeta(\vec{\tau}) = \sum_{\vec{j}=\vec{0}}^{\vec{N}}\Phi_{\vec{j}}\,K_\zeta(\vec{\tau}-\vec{\Delta}\circ\vec{j}); \quad \Phi_{\vec{0}} \equiv 0 \\ &\vec{\tau} = \vec{v} - \vec{u}; \quad \vec{\Delta}\circ\vec{k} = \{\Delta_i \cdot k_i\}_{i=1}^{N} \end{aligned} \quad (9)$$

This is a multi-dimensional vector, which is obtained as for one dimension (one-dimensional autoregressive was $\zeta(t + k\,\Delta t)$ or ζ_{t+k}), but not multiplication by a scalar and a vector but by a scalar (dot) product. The character of the numerical implementation of the Yule-Walker equation is given in Appendix 2.

For three-dimensional autoregressive models of waves (2-D space + 1-D temporal) the general model (8) is represented by (7). In this case, the equation (9) can be written as follows

$$\begin{cases} K_\zeta(p,0,..,0) = \sum_{k\neq 1}\sum_{j=1}^{N_k}\Phi_{kj}K_\zeta(p, j\circ\Delta_2,..,j\circ\Delta_N) \\ \qquad\qquad + \sum_{i=1}^{N_1}\Phi_{1i}K_\zeta(|i-p|,0,..,0) \\ K_\zeta(0,q,..,0) = \sum_{k\neq 2}\sum_{j=1}^{N_k}\Phi_{kj}K_\zeta(j\circ\Delta_1, q,..,j\circ\Delta_N) \\ \qquad\qquad + \sum_{i=1}^{N_2}\Phi_{2i}K_\zeta(0,|i-q|,..,0) \\ K_\zeta(0,0,t,..,0) = ... \end{cases} \quad (10)$$

This system of Yule-Walker equations is obtained by multiplying the values of the realization of the random field $\zeta(\vec{u})$ (where $\vec{u} = \{x, y, t\}$) and then averaging

Matrix system (10) has a coordinate-wise block structure. In this case, it involved all the values of the net covariance surface in a chosen region of space, not just the marginal values, which was used in the construction process. The matrix also is symmetric, Toeplitz and has a dominant main diagonal, which allows you to apply Cholesky decomposition to the system of equations (10).

The variance of the white noise model (7) is defined similarly for $u = v$, which leads to the relation:

$$\sigma_\varepsilon^2 = D_\zeta - \sum_{i=1}^{N}\sum_{j=1}^{N_i} \Phi_{ij} K_\zeta(\vec{\delta}_{ij});$$

$$\vec{\delta}_{ij} = \{0,..,0,i\Delta_j,0,..,0\}$$

VALIDATION OF AUTOREGRESSION MODEL

Despite the fact that the equivalence of models of Longuet-Higgins and ARMA has been proven theoretically, the problem with the application of the AR model is that it is not obvious and its apparent lack of physics of the process, in contrast to the Longuet-Higgins model. For this purpose, validation of three-dimensional model was carried out for ocean waves. For this, a series of tests was performed:

1. Simulated aerial photograph (APS)
2. Imitation of mixed ocean waves
2. Analysis of the dispersion relation
4. Simulating the evolution of ocean waves in a storm

Simulation of Aerial Photograph (APS)

As a test, we examined the visual compatibility of snapshots obtained by APS and simulations. The aim was to obtain a model snapshot, visually representing 3-D ocean waves. For a complete validation of the proposed approach based on the snapshot APS evaluated the statistical characteristics of the elements of visible waves resulting from simulation of a real case of ocean waves. For this a series of plates available from areal stereo-photogrammetry of pure wind storm waves in 1963 in the Barents Sea (Davidan, *et al.*, 1971) was considered. On the snapshot APS allocated following wave characteristics: height, length, length of wave crests, slope, and three-dimensionality (short crestedness and bidirectionality).

Based on numerous experimental data (Davidan, *et al.*, 1978) that the cumulative distribution function (CDF) of the above characteristics of the visible waves are well approximated by the Weibull distribution with an exponent *k*:

$$F(x) = \exp\left[-A\left(\frac{x}{\bar{x}}\right)^k\right]; \qquad A = \Gamma^k\left(\frac{1}{k}+1\right)$$

The parameters and their distribution coefficient for different elements of the visible waves are shown in Table 1.

It should be noted that none of these characteristics in the simulated realizations of the space-time fields of ocean waves were useed as input data. Their reproduction will be an indirect indication of the adequacy of the proposed AR model. In addition, the distribution of lengths, periods, and the steepness of the waves determines the dispersion of gravitational waves.

Table 1. Parameters of the Weibull distribution for the various elements of visible waves (Davidan, *et al.*, 1978)

Name	***k***
Wave height	2.0
Wavelengths	2.3
The lengths of the wave crests	2.3
Wave period	3.0
Wave steepness	2.5
Three-dimensionality	2.5

To estimate the goodness of fit to the Weibull distributions, it is convenient to use the method of straight line log-log diagrams (Hahn & Shapiro, 1994). For each *k*, a functional grid with a uniform step in the argument *x* has been constructed for the *y*-axis using the ratio

$$x_F = \left(-\frac{\ln F}{A}\right)^{1/k}$$

Results of statistical analysis are shown in Figs. 1–4, where the distribution of relevant parameters are given in log-log grids of the Weibull distribution with parameters *k* from Table 1 (Boukhanovsky & Degtyarev, 1996; Boukhanovsky, 1997).

Analysis of one-dimensional distribution laws does not exhaust all of the models of wave statistics. The elements of visible waves are statistically dependent, and their joint distributions cannot be represented as the product of one-dimensional marginal distributions of each of the variables. To evaluate the ability of models to adequately represent the joint distribution of the parameters of waves, an analysis of the joint CDF of heights and wavelengths was performed. Boukhanovsky

(1997) analyzed the joint CDF of heights and wavelengths, which showed very good agreement.

Typical indicators of statistical dependence of random variables are the first and second central moments of their joint distribution, curve of regression (conditional mean value)

$$m_h(\lambda) = \int_{-\infty}^{\infty} h \cdot f(h/\lambda) dh \qquad (11)$$

and skedastic curve (conditional variance):

$$\sigma_m^2(\lambda) = \int_{-\infty}^{\infty} \left(h - m_h(\lambda)\right)^2 \cdot f(h/\lambda) dh \qquad (12)$$

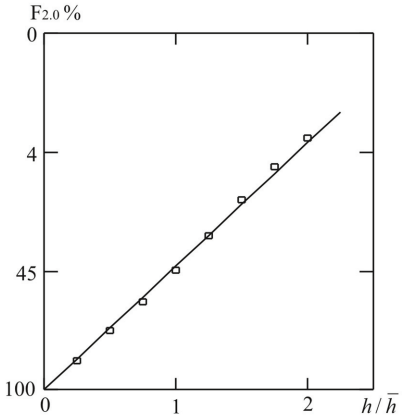

Fig. 1 Distribution of wave heights, □ – model, line – experiment.

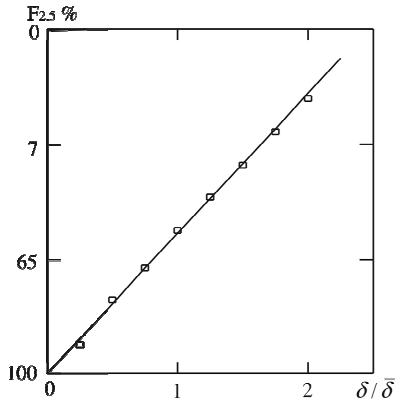

Fig.2 Distribution of wave steepness, □ – model, line – experiment.

For the sample used for estimating the joint CDF of heights and wavelengths, statistical estimates of relationships (11) and (12) were constructed. These are presented in normalized form in Figure 5 and 6. The figures also show approximations of the experimental data given in Davidan, *et al.* (1978). It should be noted that the characteristic shape of the conditional variance of wavelengths, from their heights (Fig. 6) was not achieved by the Longuet-Higgins model.

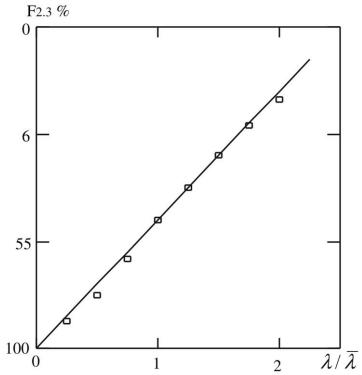

Fig.3 The distribution of wavelengths, □ – model, line – experiment.

Simulation of Mixed Ocean Waves

Simulation of measurements of wave the duration at a fixed-point were used in the validation of the model. Several options for the composition of wind waves and swell systems have been investigated. In all cases, the disturbance was assumed Gaussian. An analytical description of frequency-directional spectra of this type of disturbances is only possible for some of the simplest cases, where the different wave systems have significantly different periods and can be approximated in the form of an elementary sum of spectra.

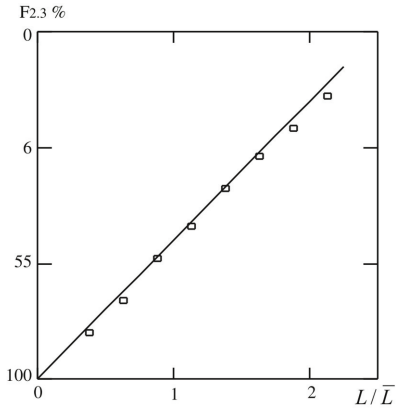

Fig.4 Distribution of the lengths of the crests, □ – model, line – experiment.

The frequency-directional spectrum of any system of waves can be represented in the form of

$$S_\zeta(\omega, \Theta) = S_\zeta(\omega) \cdot Q(\omega, \Theta)$$

where $S_\zeta(\omega)$ is the frequency spectrum of ocean waves, and $Q(\omega,\Theta)$ a function of the angular distribution of energy.

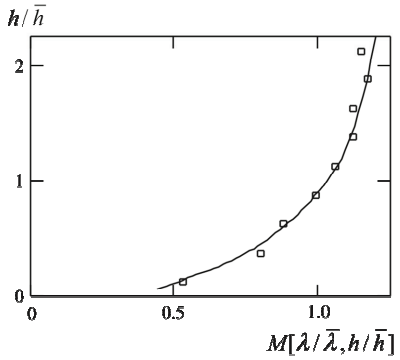

Fig. 5 Regression of wavelengths and wave heights, □ – model, line – experiment. (Boukhanovsky & Degtyarev, 1996; Boukhanovsky, 1997)

For validation of a probabilistic model of ocean waves, a number of frequency-directional spectra of mixed disturbances with one and two systems of swell have been used, based on wave measurements in the Pacific (Rozhkov, 1990) and the Atlantic (Register of the USSR, 1974) oceans. In this case, it makes sense to classify considered according to the following criteria:

- spectrum has two clearly defined peaks, similar in height.
- spectrum has two clearly defined peaks, greatly different in height.
- spectrum has a clear peak corresponding to the system of waves and a weak high-frequency perturbation of the system of wind waves.
- spectrum has three or more distinct peaks.

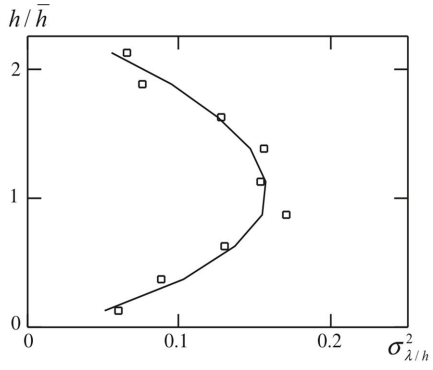

Fig.6 Skedastic curve (conditional variance) of lengths and wave heights, □ - model, line - experiment. (Boukhanovsky & Degtyarev, 1996; Boukhanovsky, 1997)

During the simulations the dependence of the angular distribution was approximated as

$$Q(\omega,\theta) = C_r \cos^r \theta$$

where $r = 4$ for wind waves; and $r = 8$ for swell.

Validation was conducted using numerical estimates of average wave heights and periods and their distributions.

Figs 7–10 show some results of validation. The spectrum of ocean waves shown in Fig. 7, corresponds to mixed waves observed in the tropical Pacific Ocean July 8, 1986. According to Rozhkov (1990) the distribution of its periods can be represented as

$$F(\tau) = aF_{ww}(\tau) + (1-a)F_s(\tau) \quad (13)$$

where, according to the analysis the wind wave period τ_{ww} is 5.0 s, the swell period τ_s is 9.0 s, and a = 0.70. Figure 8 shows the estimated CDF of periods of mixed ocean waves, its empirical estimation and approximation (13), smoothed on a log-log plot of a Weibull distribution with exponent $k = 3$.

For ordinary wind-waves the distribution of periods is well smoothed on a log-log plot of a Weibull distribution with $k = 3$, but for decreasing bandwidth of the spectrum, increases to 4. This confirms the theoretical result obtained by Yu. M. Krylov (Krylov, *et al.*, 1966).

Analysis of the Dispersion Relation

We use the high correlation of the heights and periods between the consecutive waves. This means that the change in frequency $\omega(t)$ of consecutive waves is a slowly varying process. Then a synchronous implementation can generate $\zeta(t)$, wave slope $a(t)$=arctan($d\zeta(x,t)/dt$) and the wave frequency $\omega(t)$ at a given point. Dispersion of waves is the dependence of the velocity distribution on their frequencies. This means that to study the dispersion relation it is necessary to investigate the wave propagation speed.

We can define the evolution of space coordinate X with the help of numerical integration of elementary first order differential equation $\dot{X}(t) = V(t)$ where $V(t)$ is the stochastic wave speed.

We can define stochastic wave speed according to the theory of small amplitude waves

(in reality we have small amplitude because ratio wave height/wavelength is not more than 1/20):

$$V(t) = \frac{\omega(t)}{k(\omega(t))}$$

We already have a model for the stochastic wave frequency $\omega(t)$. Now it is necessary to obtain $k(\omega(t))$.

The classical dispersion relation for progressive limit small amplitude waves is:

$$\omega = \sqrt{kg} \qquad (14)$$

So the correlation between wave amplitude spectrum and wave slope spectrum is:

$$S_\alpha(\omega) = k^2 S_\zeta(\omega) = \frac{\omega^4}{g^2} S_\zeta(\omega)$$

Analysis of wave slope spectrum and wave amplitude spectrum for sea states 5 and 8 has been carried out. Records from irregular wave tests in the towing tank of the Technical State University of Kaliningrad were used. These spectra were also used for $\zeta(t)$ and $\alpha(t)$ generation.

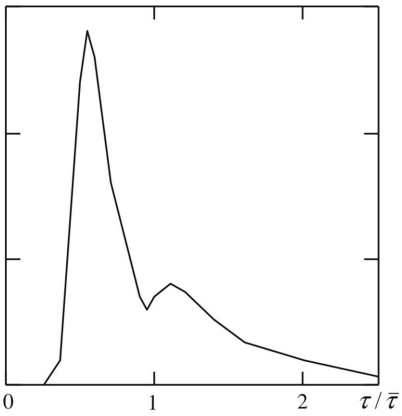

Fig.7 Spectrum of wave system with one swell component (Rozhkov, 1990)

Figure 11 shows the results of reconstruction of the dispersion relation $k(\omega)$ based on the integration of the equation for the changes with the spatial coordinates X, of three synchronous wave records (Boukhanovsky & Degtyarev 1995).

Based on the results of these thre studies, we can speak of the acceptable physical adequacy of the proposed autoregression model of ocean waves in quasi-stationary conditions.

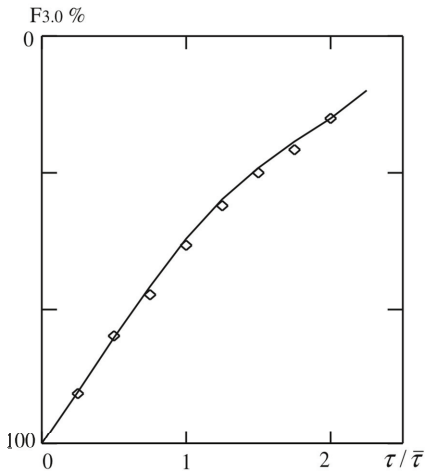

Fig.8 Distribution of wave periods, ◇ – model, line – experiment.

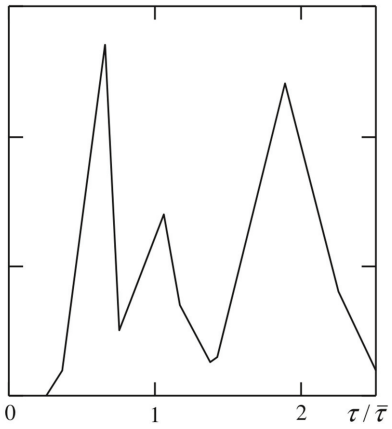

Fig.9 Spectrum of wave system with two swell components (Register of the USSR, 1974)

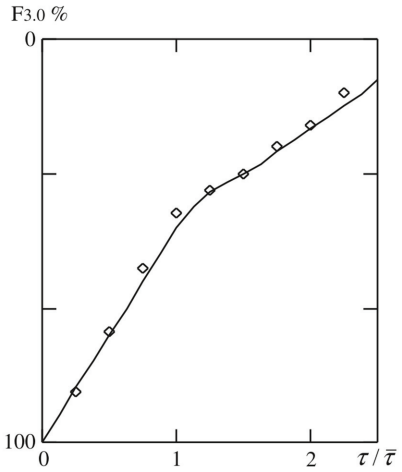

Fig.10 Distribution of wave periods, ◇ – model, line – experiment.

Simulating the Evolution of Ocean Waves in a Storm

To assess the ability of the proposed approach to simulate the wave fields under synoptic variability, we consider a number of examples of individual storms. In this case, the ARMA model (5) can be transformed as follows

$$\zeta_{\vec{v}}(t) = \sum_{\vec{j}=\vec{0}}^{\vec{N}} \Phi_{\vec{j}}(t)\zeta_{\vec{v}-\vec{j}} + \sum_{\vec{k}=\vec{0}}^{\vec{P}} \Theta_{\vec{k}}(t)\varepsilon_{\vec{v}-\vec{k}}$$

where $\Phi_j(t)$ is the change or evolutionary cycles that is determined solely by the source data. The procedure for validation of the statistical characteristics of the visible elements of the waves does not depend on the nature of non-stationarity, which allows us to check the adequacy of the model disturbances, using the simplest version of the original data.

The behavior of the wave field in a storm is provided by the example of Hurricane "Faith" in the central North Atlantic, from 2100 02.09 to 2100 06.09 1966. During this time the weather ship *Weather Adviser* served as meteorological measurements in the area of weather station "India" and made synoptic time-wave measurement.

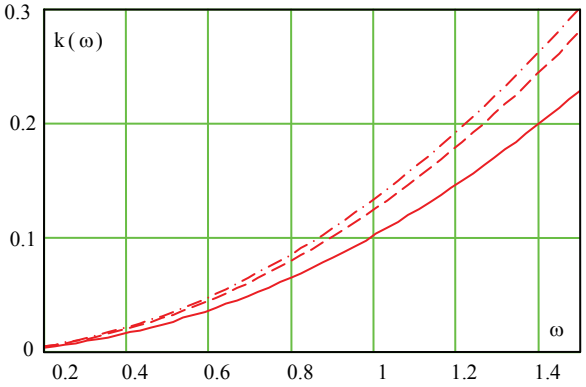

Fig. 11 Analysis of $k(\omega)$ based on results of model test in towing tank (Boukhanovsky & Degtyarev 1995), dash – sea state 5, dash dot – sea state 8, solid line – limit small amplitude waves (14).

Set of external conditions in the storm—wind direction and force do not remain constant, because ocean waves are not a quasi-stationary field. The fields of this type are usually characterized by time-frequency spectrum by taking into account the direction of $S_\zeta(\omega, \Theta, t)$. Fig. 12 presents an evaluation of time-frequency spectrum $S_\zeta(\omega, t)$, obtained from measurements of the waves in synoptic time.

The duration of the storm was divided into 32 3-hour synoptic intervals. For each, the cross section of the time-frequency spectrum was estimated from a 15-minute measurement. Figure 13 shows the evolution of the average height of waves during a storm.

In addition, the model of unsteady wave field evolution was verified on a longer time interval from 5 to 17 July 1986 using measurements in the South Pacific (Rozhkov, 1990), on the basis of records of the disturbance's time-frequency spectrum. The evolution of the disturbance was divided into 36 8-hour sections, during each interval of which the ocean waves were assumed quasi-stationary.

Both tests showed a good agreement between observations and experiments.

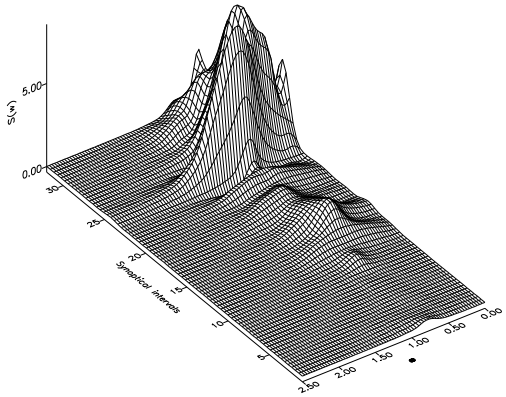

Fig.12 Time-frequency spectrum of hurricane "Faith"

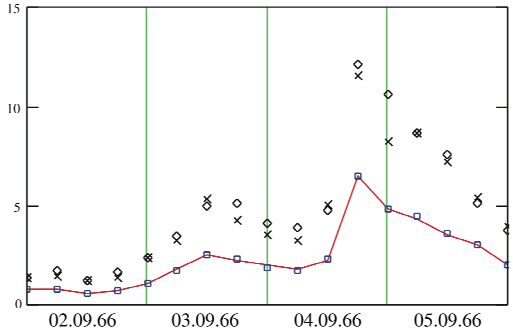

Fig.13 Diagram of average wave height variation during hurricane "Faith"; solid line - experimental estimation of average wave height; □ - model estimation of average wave height; ◇ - experimental estimation of maximum wave height; × - model estimation of maximum wave height.

CONCLUSION

An autoregression model for incident random waves has been developed that is far more computationally efficient than the Fourier series like models of St. Denis & Pearson, Rosenblatt, Sveshnikov, or Longuet-Higgins. This model is shown to be amenable to modeling the synoptic and temporal processes associated to the development and evolution of ocean waves in a storm.

It has been shown that the waves produced by the autoregression model have the correct statistical characteristics spatially and temporally to represent ocean waves—the desired wave spectra can be reproduced and the distributions of physical characteristics is correct. Although the model does not explicitly contain the physics of gravity waves, by using 2- and 3-dimensional (1- or 2-dimensions in space + time) autoregression functions based on actual wave measurements, the model even captures the dispersion relation for gravity waves.

It is possible to conclude that the autoregression model gives an accurate, hydrodynamically valid description of wind waves. This means that we can use the model to solve the problem of predicting the velocity potential in the fluid below the wave surface. In this case we can reduce this complicated nonlinear hydrodynamic problem with an unknown boundary to linear problem with a known boundary and linear boundary conditions.

It remains to be seen if the physics of gravity waves can be incorporated in the autoregressive model explicitly. This would be critical to representing idealized spectra without requiring the generation of the waves experimentally or by advanced numerical simulation so that the 2- or 3-D autocorrelation function can be computed.

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APPENDIX 1—DERIVATION OF THE AUTOREGRESSION COEFFICIENTS FOR A 1-DOF SYSTEM

Employing Eqn. (6), the coefficients of the autoregression model can be determined from the autocorrelation function. The following outlines the process for determining the necessary coefficients.

Start by multiplying both sides of (6) by ζ_t yielding:

$$\zeta_t^2 = \zeta_t \sum_{i=1}^{N} \Phi_i \zeta_{t-i} + \zeta_t \varepsilon_t \,,$$

and, integrate both sides from 0 to T multiplying by $1/T$:

$$\frac{1}{T}\int_0^T dt\, \zeta_t^2 = \frac{1}{T}\int_0^T dt\ \zeta_t \sum_{i=1}^{N} \Phi_i \xi_{t-i} + \frac{1}{T}\int_0^T dt\, \zeta_t \varepsilon_t \,.$$

In the limit as $T \to \infty$, the integral on the left-hand-side becomes the variance, V_ζ, of ζ_t, and the second integral on the right-hand side goes to zero, leaving:

$$V_\zeta = \frac{1}{T}\int_0^T dt\ \zeta_t \sum_{i=1}^{N} \Phi_i \zeta_{t-i} \,.$$

Rearranging the order of integration and summation gives:

$$V_\zeta = \sum_{i=1}^{N} \Phi_i \frac{1}{T}\int_0^T dt\ \zeta_t \zeta_{t-i} \,,$$

where, in the limit as T $\to\infty$, $\frac{1}{T}\int_0^T dt\ \xi_t \xi_{t-i}$ becomes the autocorrelation function K_ζ with lag k.

$$V_\zeta = \sum_{i=1}^{N} \Phi_i K_\zeta(k) \,. \qquad \text{(A1.1)}$$

Equation (A1.1) is the first of a series of linear equations for the unknown coefficients of the autoregression model for ζ_t.

To develop the second equation, multiply both sides of (6) by ζ_{t-1}, resulting in:

$$\zeta_{t-1}\zeta_t = \zeta_{t-1} \sum_{i=1}^{N} \Phi_i \zeta_{t-i} + \zeta_{t-1}\varepsilon_t$$

and, as with the first equation, integrate both sides from 0 to T multiplying by $1/T$:

$$\frac{1}{T}\int_0^T dt\, \zeta_{t-1}\zeta_t = \frac{1}{T}\int_0^T dt\, \zeta_{t-1} \sum_{i=1}^{N} \Phi_i \zeta_{t-i} + \frac{1}{T}\int_0^T dt\, \zeta_{t-1}\varepsilon_t .$$

In the limit as $T \to \infty$ the integral on the left-hand-side becomes the autocorrelation function of ζ_t with lag 1, $K_\zeta(1)$; and the second integral on the right-hand side goes to zero, leaving:

$$K_\zeta(1) = \frac{1}{T}\int_0^T dt\, \zeta_{t-1} \sum_{i=1}^{N} \Phi_i \zeta_{t-i} \,.$$

Rearranging the order of integration and summation gives:

$$K_\zeta(1) = \sum_{i=1}^{N} \Phi_i \frac{1}{T} \int_0^T dt\, \zeta_{t-1}\zeta_{t-i},$$

where in the limit, $\frac{1}{T}\int_0^T dt\ \zeta_{t-1}\zeta_{t-i}$ becomes the autocorrelation function K_ζ with lag $k-1$,

$$K_\zeta(1) = \sum_{i=1}^{N} \Phi_i K_\zeta(k-1). \qquad \text{(A1.2)}$$

Equation (A1.2) is the second in a series of linear equations for the unknown coefficients of the autoregression model for ζ_t.

Generalizing this, multiply (6) by ζ_{t-n} and follow the same procedure used in deriving the first two linear equations, resulting in:

$$K_\zeta(n) = \sum_{i=1}^{N} \Phi_i K_\zeta(k-n) \quad \text{(A1.3)}$$

Recognizing that the variance is the autocorrelation function with a lag of zero, $V_\zeta = K_\zeta(0)$, Eqn.(A1.3) is valid for $n = 0, 1, 2, \ldots$.

Thus, we have a general formula for defining the system of linear equations for the unknowns F_k, $k = 1 \ldots, N$ needed to define the autoregression model for ζ_t of order N, Eqn. (6).

These equations for the coefficients of the autoregression equation are known as the Yule-Walker equations (Box, *et al.*, 2008). They are named after G. Udny Yule and George Walker who developed the first autoregressive models in the late 1920's and early 1930's (Yule, 1927; Walker, 1931).

Writing the one-dimensional Yule-Walker equations compact notation, we obtain

$$K_\zeta(\vec{\tau}) = \sum_{\vec{j}=\vec{0}}^{\vec{N}} \Phi_{\vec{j}}\, K_\zeta(\vec{\tau} - \vec{\Delta} \circ \vec{j}) + \sum_{\vec{k}=\vec{0}}^{\vec{P}} \Theta_{\vec{k}}\, K_{\zeta\varepsilon}(\vec{\tau} - \vec{\Delta} \circ \vec{k})$$

$$\vec{\tau} = \vec{v} - \vec{u}\,; \qquad \vec{\Delta} \circ \vec{k} = \left\{\Delta_i \cdot k_i\right\}_{i=1}^{N}$$

or

$$K_\zeta(\vec{\tau}) = \sum_{\vec{j}=\vec{0}}^{\vec{N}} \Phi_{\vec{j}} K_\zeta(\vec{\tau} - \vec{\Delta} \circ \vec{j}); \qquad \Phi_{\vec{0}} \equiv 0$$

$$\vec{\tau} = \vec{v} - \vec{u}; \vec{\Delta} \circ \vec{k} = \left\{\Delta_i \cdot k_i\right\}_{i=1}^{N}$$

It is clear that we have to use the ACF of real data.

APPENDIX 2—NUMERICAL SOLUTION OF THE YULE-WALKER EQUATIONS

The matrix of the generalized Yule-Walker system has the following properties:

- Matrix is symmetric and positive semidefinite. This quality is a trivial consequence of non-negative definite quadratic form describing any covariance surface.
- Matrix is Toeplitz, ie, $A_{i+1,j+1} = A_{ij}$. This follows from the principle of building the system, because

  $$A_{ii} = K_\zeta(\vec{0}), \quad \forall i = 1, 2, \ldots$$

- Matrix has a dominant main diagonal. Indeed, for the covariance surfaces of stationary random fields must satisfy:

  $$A_{ii} = K_\zeta(\vec{0}) \geq K_\zeta(\vec{u}) = A_{ij}, \quad \forall i \neq j, \forall \vec{u} \neq \vec{0}$$

To solve the system of equations with this matrix, the method for square roots (Cholesky decomposition) is proposed. It belongs to the class of direct methods for solving algebraic systems and is based on the decomposition of a symmetric matrix as the product of two triangular matricies:

$$A \equiv L \times L^T,$$

whose coefficients have the following relationships (straight run):

$$l_{11} = \sqrt{a_{11}}\,; \qquad l_{j1} = a_{j1} \Big/ l_{11}\,; \qquad j = \overline{1, M}$$

$$l_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} l_{ik}^2}\,; \quad l_{ji} = \frac{a_{ji} - \sum_{k=1}^{i-1} l_{ik} l_{jk}}{l_{ii}}\,; \quad i = \overline{1, j}$$

The reversal of the method is performed identically to the method of Gauss elimination.

Compared with the typical representatives of a class of direct methods (Gauss elimination, LU-decomposition method, etc.). This approach has the following advantages:

- Cholesky decomposition requires approximately half the computational cost of the direct approaches.
- Cholesky decomposition requires approximately half the amount of storage.
- Cholesky decomposition is guaranteed stable against the error in the initial data.

It is worth noting that for a generalized Yule-Walker method, Cholesky decomposition may be used with any of the direct methods, which use pivoting, which is governed by the predominance of the main diagonal of the matrix.

Unfortunately, the Levinson-Durbin approach well-researched and adapted for solving the one-dimensional autoregressive method of is not as effective in solving the generalized Yule-Walker system.

Indeed, in the one-unit increase in the order of autoregression leads to a halo of the original matrix, i.e. increases its order by 1, that allows us to construct a recursion procedure for estimating the parameters. However, in the multidimensional case, the increase of the order of 1 in the *i*-th coordinate leads to the bordering of a matrix of the generalized Yule-Walker system of $\prod_{i \neq j} N_j$

rows and columns, which increases the dimension of space, and coordinate-wise order autoregression leads to wasted computational cost.

The variance of white noise, σ_ε^2 is determined from the system of equations (8) with $u = v$:

$$\sigma_\varepsilon^2 = D_\zeta - \sum_{\vec{j}=\vec{0}}^{\vec{N}} \Phi_{\vec{j}} K_\zeta(\vec{j} \circ \vec{\Delta}) \qquad \text{(A2.1)}$$

Many authors have noted an excessive sensitivity of the one-dimensional autoregressive model to errors such as red and white noise in the raw data. By analogy with the one-dimensional case, to reduce this oversensitivity, you can use an overdetermined generalized Yule-Walker system, minimizing the error of the solution in the quadratic norm using the method of least squares (Maddala, 1971):

$$\Phi = (A^T A)^{-1} A^T B ,$$

where A is a rectangular matrix PxN, $P > N$.

In this case, the zero time autoregressive spectrum may not coincide with the variance of the simulated field, which forces us to use the spectral ratio:

$$D_\zeta = \int_0^{\pi/\Delta_1} ..(N).. \int_0^{\pi/\Delta_N} S_\zeta(\vec{\omega}) \prod_{k=1}^{N} d\omega_k$$

to model the multivariate autoregressive variance:

$$\sigma_\varepsilon^2 = \frac{\pi^N K_\zeta(\vec{0})}{J};$$

$$J = \int_0^{\pi/\Delta_1} ..(N).. \int_0^{\pi/\Delta_N} \frac{\prod_{k=1}^{N} \Delta_k \, d\omega_k}{\left| \sum_{\vec{j}=\vec{0}}^{\vec{N}} \Phi_{\vec{j}} \exp\left[i \sum_{k=1}^{N} \left(j_k \Delta_k \omega_k \right) \right] - 1 \right|^2}.$$

It is worth noting that the calculation of multidimensional integrals is a very time consuming task, especially for high order coordinate-wise autoregression. In recognition of this, the most effective way of identifying patterns in random fields should use a fully defined system of Yule-Walker relations (A2.1) in the presence of smooth (approximate) initial data.