Evaluation of the critical wave groups method for calculating the probability of extreme ship responses in beam seas

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ABSTRACT

The paper investigates the accuracy of the current formulation of the "critical wave groups" method for calculating the probability of extreme responses of vessels rolling in beam seas. The method employs short duration regular excitations to identify "critical" for ship stability wave events that cause slight exceedance of a given roll angle threshold. The probability of any exceedance of the roll angle threshold is then estimated by the probability of encountering any wave sequence higher than the determined critical, based on wave height and period distributions derived from spectral methods. In this study the "critical wave groups" method is extended by incorporating realistic wave group forms, characterized by high probability of occurence. Both the regular and the irregular wave group schemes are applied to evaluate the probability of exceedance for several roll angle thresholds for two ship models. To increase the accuracy of the approach, wave group statistics are obtained from direct simulations of the wave field rather than from spectral methods. The results are tested against Monte Carlo simulations of ship roll motion.

Keywords: *wave group, probability, instability, roll, dynamics, resonance, rare events.*

1. INTRODUCTION

The study of large amplitude ship roll motions in stochastic beam seas is a non-trivial task expanding in both the fields of non-linear dynamics and probability. As known, roll statistics deviate from Gaussianity with increasing level of non-linearity, leading to probability distributions with heavy-tailed structure (Belenky et al., 2016b). However, calculating the probability of extreme roll events by employing "brute force" methods suffers from a number of deficiencies. First, the accuracy of a "direct counting" definition of probability becomes questionable when dealing with rare events. At the same time, the fact that ship response is not essentially an ergodic random process in the case of a non-linear system further increases the computational burden for tracing the complex shape of the tails (Belenky et al., 1998).

Several methods have been proposed to treat the so called "problem of rarity", described in the above. Extrapolation methods employ statistics based on a limited number of realizations to predict the probability of an event that is too rare to be observed. The concept derives from Extreme Value Theory which provides asymptotic expressions for the distribution of the maximum of a sample of independent and identically distributed random variables. Thus, the objective is the estimation of the parameters of an extreme value distribution by fitting the latter to a set of experimental or simulation data. The method has been demonstrated in several studies and much effort has been put into addressing practical issues regarding its application for ship stability assessment (e.g., Belenky et al., 2016a; Campbell et al., 2016).

On the other hand, wave group methods offer an alternative solution to the problem by focusing on specific time intervals when dangerous wave events occur. One of them is the "critical wave groups" method which quantifies instability tendency through the probability of encountering any wave group that could have provoked the instability (Themelis and Spyrou, 2007). In the deterministic part of the method, regular wave trains are employed to identify critical, in terms of ship stability, height thresholds. Then, in the probabilistic part, the probability of encountering any wave sequence higher than the specified thresholds is calculated using distributions of wave heights and periods derived from spectral methods. A first attempt to validate the concept was presented by Shigunov et al. (2012) who selected a modern 8000 TEU containership to calculate the probability of exceedance for a 40degrees roll angle threshold. The results were tested against Monte Carlo simulations and fair coincidence was noted in the case of beam seas excitation.

As a next step, in this paper we employ the "critical wave groups" method to predict the probability of exceedance for a number of roll angle thresholds for two different ship models. At the same time, our recent work towards improving the determistic part of the approach is continued, by incorporating more realistic wave group forms. The idea is to identify critical wave events in terms of the "most expected" wave groups of a given sea state using the method developed by Anastopoulos et al. (2016). To eliminate the impact of spectral methods on the accuracy of the probabilistic part, desired height and period distributions are obtained from direct simulations of the wave field. Finally, the conditions under which the "critical wave groups" method produces comparable results with those obtained from Monte Carlo simulations of roll motion are investigated and the focus is set on the region of extreme responses where the accuracy of the latter is disputable.

2. MATHEMATICAL FORMULATION

In the field of ocean and coastal engineering, wave groups are traditionally considered as sequences of waves with heights exceeding a certain preset level and slightly varying periods (Masson and Chandler, 1993; Ochi, 1998). Despite that several threshold-based definitions have been utilized in the past to study wave groupiness measures, one would argue that, from ship dynamics perspective, wave groups are sequences of waves which are sufficiently high to provoke instabilities.

Now, let us assume that we are interested in estimating the probability that a vessel exceeds a roll angle threshold φ_{crit}. The key idea of the "critical wave groups" method is to first identify the wave events that cause the exceedance and then, calculate the probability of encountering them. The essence of the approach is presented below:

$$p\left[\varphi > \varphi_{crit}\right] = \sum_k \underbrace{p\left[\varphi > \varphi_{crit} \middle| \left(\bigcup_i wg_{k,i}, ic_k\right)\right]}_{=1} \times p\left[\bigcup_i wg_{k,i}, ic_k\right] \tag{1}$$

where $wg_{k,i}$ is a wave group event with characteristics i, determined for the k^{th} set of initial conditions $\{\varphi_0, \dot{\varphi}_0\}$ of the vessel at the moment of the encounter. From a preliminary investigation, Themelis and Spyrou (2008) concluded that for sea states of moderate severity the influence of initial conditions may not be very significant and thus, examining only the upright position of the vessel $\{\varphi_0, \dot{\varphi}_0\} = \{0,0\}$, denoted by $k=0$, can be somehow acceptable:

$$p\left[\varphi > \varphi_{crit}\right] = \sum_k p\left[\bigcup_i wg_{k,i} \middle| ic_k\right] \times p\left[\, ic_k\right] \approx p\left[\bigcup_i wg_{0,i} \middle| ic_0\right] \tag{2}$$

Eventually, the method is implemented in two parts: a purely deterministic one, focused on the identification of the so called "critical" wave groups, i.e., those wave successions leading to only slight exceedance of φ_{crit}; and a probabilistic part to calculate the probability of encountering any wave group higher than the determined critical. As realized, the accuracy of the method depends explicitly on the shape of the critical wave groups which are in fact height thresholds for the wave events that result in $\varphi > \varphi_{crit}$.

By assuming that individual wave group occurrences are independent events, eq. (2) is reformulated as (Themelis and Spyrou, 2007):

$$p\left[\varphi > \varphi_{crit}\right] = \sum_{i=1}^{N} p\left[wg_{0,i}\right] - \sum_{i=1}^{N-1} \sum_{r=i+1}^{N} p\left[wg_{0,i}, wg_{0,r}\right] + \cdots + (-1)^{N-1} p\left[wg_{0,1}, wg_{0,2}, \ldots, wg_{0,N}\right] \tag{3}$$

where $p\left[wg_{0,1}, wg_{0,2}, \ldots, wg_{0,N}\right] = \prod_{i=1}^{N} wg_{0,i}$.

A significant challenge in eq. (3) is to ensure that wave groups that provoke exceedance of φ_{crit} form a set of mutually exclusive and collectively exhaustive events. To avoid possible overlaps in the calculations, it is convenient to identify wave groups with respect to their run length j, which is the number of consecutive heights exceeding a critical threshold:

$$p\left[wg_{0,i}\right]=p\left[\bigcup_{m}\left\{j=i,\mathbf{H}_i>\mathbf{h}_{cr,i},\mathbf{T}_i\in \mathrm{T}_{cr,m}\right\}\right] \quad (4)$$

where $\mathbf{H}_i=\{H_1,...,H_i\}$ and $\mathbf{T}_i=\{T_1,...,T_i\}$ are vectors of random variables referring respectively to the heights H_n and periods T_n of an individual wave group event with run length i $(1\le n\le i)$, $\mathbf{h}_{cr,i}=\{h_{cr,1},...,h_{cr,i}\}$ is a deterministic vector for the heights of a critical wave group with run length i and $\mathrm{T}_{\mathrm{cr},m}$ is the m^{th} range within which the critical periods are considered to vary. In the case of regular wave groups the width T_{w} of all critical ranges $\mathrm{T}_{\mathrm{cr},m}$ $(m=1,2,...,M)$ is fixed.

Modelling of wave successions as Markov chains has been one of the most successful approaches in wave group theory. Kimura (1980) was the first to elaborate on wave group statistics assuming that wave heights and related periods are Markov processes. Ever since the concept has been tested several times against numerical simulations and real wave field measurements with remarkable success (e.g., Stansell et al., 2002). In this context, the probability of encountering dangerous wave groups with certain specifications, as in eq. (4), is expressed as:

$$p\left[wg_{0,i}\right]= p_0\times\prod_{n=2}^{i}\int_{h_{cr,n}}^{+\infty}\int_{\mathrm{T}_{cr,m}} f_{H_n,T_n|H_{n-1},T_{n-1}}\left(h_n,t_n|h_{n-1},t_{n-1}\right)dt_n dh_n \quad (5)$$

where $m=1,2,...,M$ denotes different cases of critical period segments and:

$$p_0=\int_{h_{cr,1}}^{+\infty}\int_{\mathrm{T}_{cr,m}} f_{H_1,T_1}\left(h_1,t_1\right)dt_1 dh_1 \quad (6)$$

In the above, $f_{H_n,T_n|H_{n-1},T_{n-1}}$ is the conditional probability density function (PDF) of two consecutive wave heights and related periods and f_{H_1,T_1} is the joint PDF of the height and period of a single wave.

Equation of roll motion

In this study ship motion is modelled under the Froude-Krylov assumption using the following simple uncoupled equation, written in terms of the relative roll angle φ:

$$\left(I_{44}+A_{44}\right)\ddot{\varphi}+D\left(\dot{\varphi}\right)+g\Delta GZ\left(\varphi\right)=M\left(t\right) \quad (7)$$

with I_{44} and A_{44} being the roll moment of inertia and the added moment of inertia, respectively, Δ is the ship displacement, g is the gravitational acceleration and D is the damping moment:

$$D\left(\dot{\varphi}\right)=B_1\dot{\varphi}+B_2\dot{\varphi}\left|\dot{\varphi}\right| \quad (8)$$

The restoring arm in still water is given as:

$$GZ\left(\varphi\right)=\sum_k C_k\varphi^k \ , \quad k=1,3,5,... \quad (9)$$

When information about the roll response amplitude operator (RAO) is available, the wave induced moment is estimated from:

$$S_{MM}\left(\omega\right)=\left|RAO\left(\omega\right)\right|^2 S_{\eta\eta}\left(\omega\right) \quad (10)$$

where $S_{\eta\eta}$ is the energy spectrum of the water surface elevation which is a stationary ergodic Gaussian process. Alternatively, in the presence of long incident waves, the concept of instantaneous wave slope at the middle of the ship α can be employed (Wright and Marshfield, 1980):

$$M\left(t\right)=-I_{44}\ddot{\alpha}\left(t\right) \quad (11)$$

Dividing eq. (7) by $I_{44}+A_{44}$ we finally obtain:

$$\ddot{\varphi}+b_1\dot{\varphi}+b_2\dot{\varphi}\left|\dot{\varphi}\right|+\sum_k c_k\varphi^k=m\left(t\right) \quad (12)$$

Construction of realistic wave groups

Anastopoulos et al. (2016) extended the Markovian model of Kimura (1980) to develop a method for the systematic construction of irregular wave group profiles, characterized by high probability of occurrence. The key is to select the height H_c and period T_c of the highest wave of the group to initiate the following iterative scheme:

$$\overline{h_i} = \int_0^\infty h_i f_{H_i|H_{i-1},T_{i-1}}\left(h_i|h_{i-1},t_{i-1}\right)dh_i \tag{13}$$

$$\overline{t_i} = \int_0^\infty t_i f_{T_i|H_i,H_{i-1},T_{i-1}}(t_i \mid h_i, h_{i-1}, t_{i-1})dt_i \tag{14}$$

Now, let us assume that we are interested in generating a sequence of j wave group heights and related periods with H_c and T_c occupying the i^{th} position $(1 \le i \le j)$. Forward application of eqs. (13) and (14) will provide the heights and periods of the waves succeeding the initial (highest) one. Then, the "most expected" past outcomes are identified by applying the same procedure backwards in time. The calculation of the conditional expectation in eq. (13) precedes that of eq. (14) so as to take into account the correlation between the height and period of a predicted wave. The transition PDFs can be obtained either from spectral methods (Anastopoulos et al., 2016) or by analyzing data collected from Monte Carlo simulations of the wave field (Anastopoulos and Spyrou, 2016).

The next step is to construct the continuous-time counterparts of the generated sequences. To this end, we opt for a representation of water surface elevation η of the form:

$$\eta(x,t) = \sum_{n=0}^{(5j-3)/2} a_n f_n(x,t) \tag{15}$$

In our earlier studies the f_n basis functions were derived from the application of the Karhunen-Loève theorem (Sclavounos, 2012). Here, aiming at reducing the computational cost related to the solution of the Karhunen-Loève eigen-problem, we employ the widely used Fourier basis functions. The number of terms kept in eq. (15) is selected so as to satisfy a set of geometrical constraints which ensure that the shape of the produced waveform is compatible with the predictions of eqs. (13) and (14). More details can be found elsewhere (e.g., Anastopoulos and Spyrou, 2016). It is noted however that the truncation order in eq. (15) is lower than the originally recommended $(6j)$ since it was recently observed that fewer terms were enough to generate desired waveforms.

3. RESULTS AND DISCUSSION

In this section, the "critical wave groups method" is applied to two different ship models in order to predict the probability of exceedance for several roll angle thresholds. Both regular and irregular wave group excitations are employed and the results are tested against Monte Carlo simulations of roll motion. To improve the overall accuracy of the approach, the PDFs of successive wave heights and periods appearing in eqs. (5) and (6) are computed from direct simulations of the water surface displacement instead of spectral methods.

Regarding the construction of irregular wave group shapes, the transition probabilities in eqs. (13) and (14) were calculated according to the method described in Anastopoulos et al. (2016) with the only difference that the necessary correlation parameters were estimated from the generated wave data. In this way, the efficiency of the Markov model for determining the "most expected" wave height and period sequences is enhanced.

Ship model 1

An ocean surveillance ship, referred in the study of Su (2012), was selected as the first ship model. Main parameters of the vessel are given in Table 1.

Table 1: Main parameters of the ocean surveillance vessel.

Parameter	Dimensional value
$I_{44} + A_{44}$	5.540×10^7 kg·m^2
Δ	2.056×10^6 kg
b_1	0.095 s^{-1}
b_2	0.052
c_1	1.153 s^{-2}
c_3	–0.915 s^{-2}

The ship is assumed to operate in a sea state described by the modified Pierson-Moskowitz (PM) spectrum with significant wave height $H_s = 4$m and peak period $T_p = 6$s:

$$S_{\eta\eta}(\omega) = \frac{5.058 g^2 H_s^2}{\omega^5 T_p^4} \exp\left[-\frac{5}{4}\cdot\left(\frac{\omega_p}{\omega}\right)^4\right] \tag{16}$$

were ω_p is the peak frequency. The wave induced moment is modelled using eq. (10) and the roll response amplitude operator $|RAO(\omega)|$ of the vessel is presented in Figure 1.

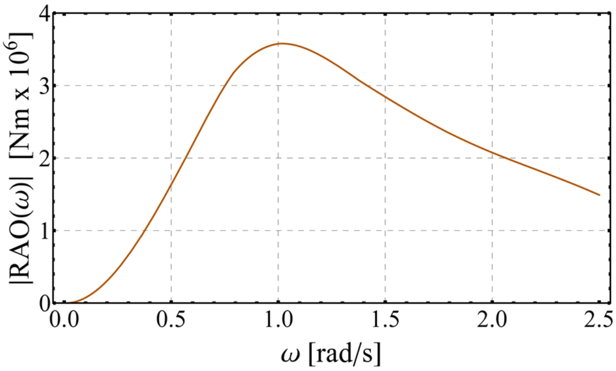

Figure 1: Roll response amplitude operator $|\text{RAO}(\omega)|$ for ship model 1.

For the simulations of the wave field, the model of Longuet-Higgins (1952) was adopted:

$$\eta(t) = \sum_n \sqrt{2S_{\eta\eta}(\omega_n)d\omega_n} \cos(\omega_n t + \varepsilon_n) \qquad (17)$$

were ε_n are random variables uniformly distributed over $[0,2\pi)$, ω_n are the frequencies of the wave components and $d\omega$ is the frequency resolution. In total, 18853 waves were analyzed from a set of 24 records of 1 hour. Finally, statistics of roll motion were estimated without assuming the ergodic property for the response (Belenky et al., 1998). As a corollary, the analysis was performed on a collection of approximately $15 \cdot 10^5$ short-duration realizations, sampled at a fixed time instant $t_s = 150\text{s}$.

In Figure 2 the iterative scheme of eqs. (13) and (14) is applied in order to predict the characteristics of the "most expected" wave groups of the examined sea state for various cases of $\{H_c, T_c\}$, values, here denoted by red nodes. The vertical axis shows the heights that derive from successive iterations and the horizontal axis shows the corresponding periods. The evolution of the procedure for a given set of $\{H_c, T_c\}$ parameters is indicated by black crosses along the dashed lines. The root of this tree-shaped diagram is the stationary state of the Markovian system and the structure of the "most expected" wave groups depends on the distance of the highest wave from the root.

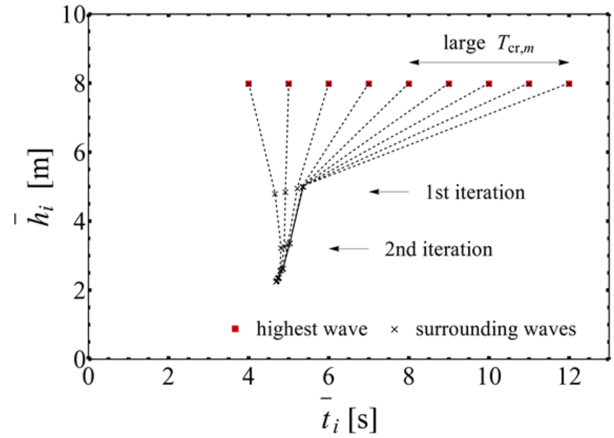

Figure 2: Characteristics of the most expected height and period sequences generated for the PM spectrum.

In Figure 3 the results of the Monte Carlo simulations (MC sim.) are presented in the same plot with the estimates of the "critical wave groups" method using regular wave groups with $j \leq 6$. For the latter two different cases of critical period range widths T_w were studied. As illustrated, for roll angles below 40degrees the method consistently underestimates the probability of exceedance. This demonstrates that for intermediate roll angle thresholds it is rather unlikely that the exceedance has been provoked by wave grouping phenomena. For larger angles, however, the accuracy of the method is improved but it is sensitive to the selection of T_w. The reason is that T_w is actually a measure of tolerance for the detection of resonant phenomena and as realized, the condition that $T_w = 1\text{s}$ is possibly too strict.

On the other hand, the method performs better for roll angle thresholds before the tail region in the case of irregular wave groups, as shown in Figure 3. In this implementation, however, the method is sensitive to the maximum period of the highest wave $T_{c,\max}$. The reason is that, for irregular wave groups, the critical period ranges $T_{cr,m}$ are defined as the difference of the shortest from the longest period encountered within a generated sequence. As shown in Figure 2, for increasing T_c the highest wave progressively deviates from the mean period of the wave group and the critical period ranges $T_{cr,m}$ become larger. Therefore, the tolerance for the detection of resonant phenomena is relaxed and the method overestimates the probability of exceedance. However, it is not clear at the moment if such cases should be included in the probability calculations since the period of the highest wave distorts the grouping character of the rest period sequence.

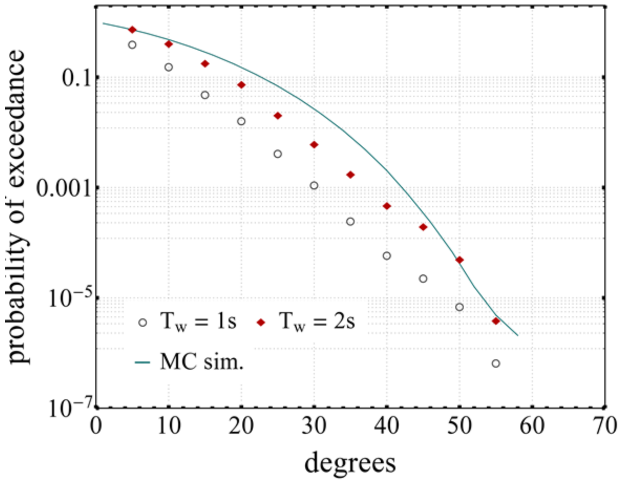

Figure 3: Probability of exceedance for ship model 1 using regular wave groups.

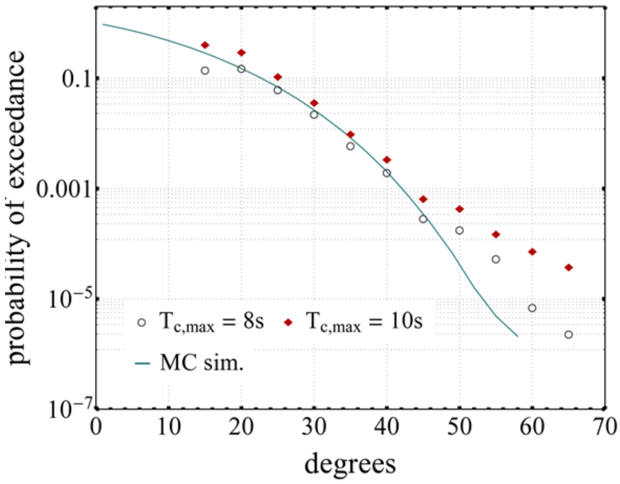

Figure 4: Probability of exceedance for ship model 1 using irregular wave groups.

In the deterministic part of the method, critical wave group parameters, identified for $\varphi_{crit} = 45^0$, are summarized in Figure 5 in the form Transient capsize diagrams. These are plots of the wave steepness of a critical wave group against its period, here normalized with the natural period of the vessel $T_o = 5.9\text{s}$ (Rainey and Thompson, 1991). Regular wave groups are given by long dashed curves while irregular wave groups are represented both by their mean steepness (short dashed line) and by the steepness of the highest wave (solid line), always against the normalized period of the latter. As one obtains two boundary lines (depending on whether he employs the mean or the maximum wave group steepness), for the case of irregular wave groups, shading has been applied between the two lines in order to enhance the contrast against the regular-wave-groups line. For $j = 2$, height thresholds defined by regular and irregular wave groups are, in the mean sense, relatively close. The shift of instability region towards the area of long waves has already been reported in Anastopoulos and Spyrou (2016). However, for $j = 3$ the dangerous zone is enlarged for the case of irregular wave groups.

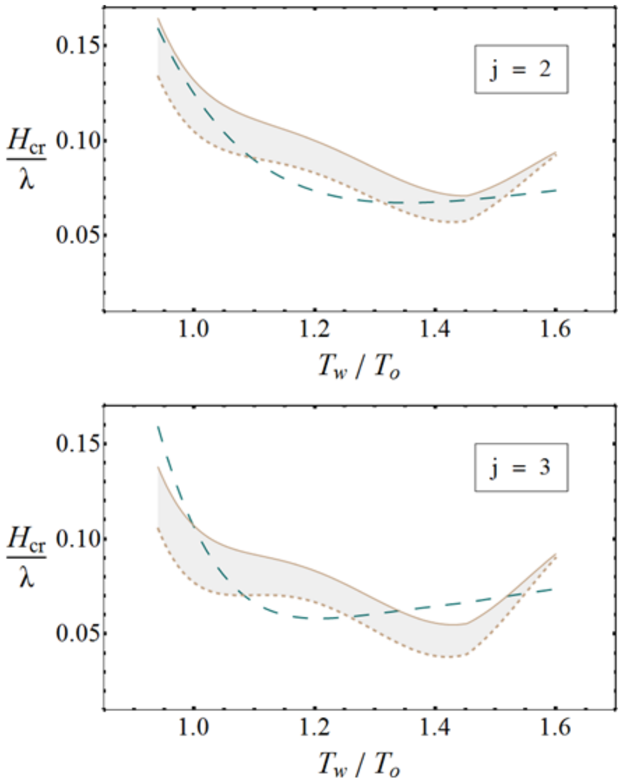

Figure 5: Transient capsize diagrams for ship model 1 for different run lengths j and $\varphi_{crit} = 45\text{deg}$.

Ship model 2

A modern 4800 TEU Panamax containership with parameters listed in Table 2 and natural period $T_o = 15.2\text{s}$ is the second ship model that was studied. The restoring arm coefficients in eq. (9) were provided directly from the loading manual of the vessel. Since no information was available about the RAO function, wave excitation was approximated by eq. (11).

In this application the JONSWAP spectrum, given in eq. (18), with parameters $H_s = 10\text{m}$, $T_p = 14\text{s}$ and $\gamma = 1.932$ was selected to describe the sea state of operation. In the same spirit, 24 records of 1 hour length were generated according to eq. (17), corresponding to a total population of 7875 waves. Monte Carlo simulations of roll motion were performed with the same setup as for ship model 1, however sampled at $t_s = 200\text{s}$.

$$S_{\eta\eta}(\omega) = \frac{0.01g^2}{\omega^5}\exp\left[-\frac{5}{4}\cdot\left(\frac{\omega_p}{\omega}\right)^4\right]\gamma^{\exp\left[-\frac{1}{2}\left(\frac{\omega-\omega_p}{0.08\omega_p}\right)^2\right]} \quad (18)$$

Table 2: Main parameters of the Panamax containership.

Parameter	Dimensional value
$I_{44}+A_{44}$	1.122×10^{10} kg·m^2
Δ	6.820×10^{7} kg
b_1	0.043 s^{-1}
b_2	0.056
c_1	1.667 s^{-2}
c_3	3.161 s^{-2}
c_5	-10.634 s^{-2}
c_7	8.349 s^{-2}
c_9	-2.150 s^{-2}

The results obtained from the implementation of the “critical wave groups” method when ship model 2 is excited by regular and irregular wave groups is shown in Figures 6 and 7, respectively. Again, for intermediate angle thresholds, better predictions are achieved by irregular waveforms. In the tail region, direct simulations of roll motion (MC sim.) fail to predict exceedances due to the problem of rarity while, in the same range, both schemes of the “critical wave groups” method yield reliable estimates.

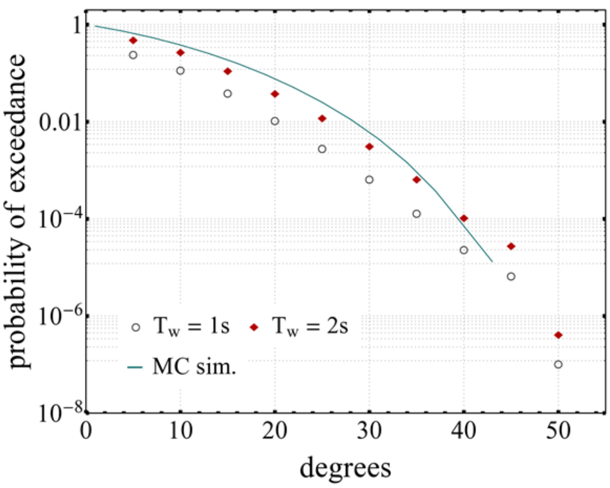

Figure 6: Probability of exceedance for ship model 2 using regular wave groups.

Finally, Figure 8 compares regular and irregular critical wave groups with run lengths $j = 2$ and $j = 3$ in terms of their individual probability of exceedance P_j. The calculations were made for the critical period parameters that provided the best agreement with the simulation results according to Figures 6 and 7. Thus, $T_w = 2\text{s}$ and $T_{c,max} = 15\text{s}$ were selected for the regular and the irregular case, respectively. The contribution of run lengths with $j > 6$ to the total probability of exceedance was found negligible.

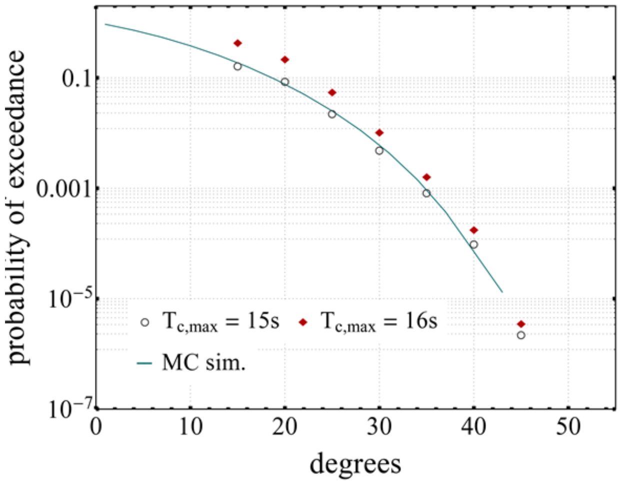

Figure 7: Probability of exceedance for ship model 2 using irregular wave groups.

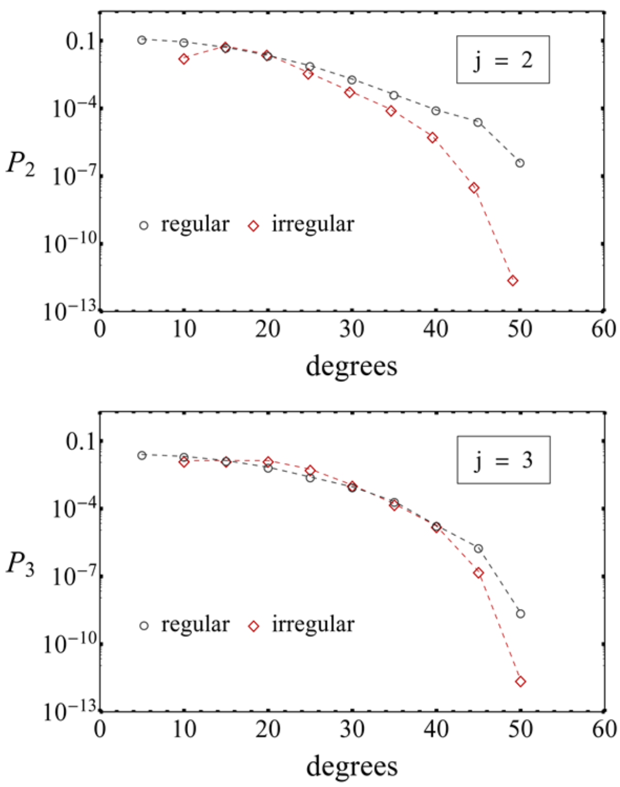

Figure 8: Contribution of individual run lengths j to the probability of exceedance for ship model 2.

4. CONCLUDING REMARKS

In this study the "critical wave groups" method was applied to predict the probability of large-amplitude ship motions in beam seas. The method was extended by incorporating realistic wave excitations representing the "most expected" wave groups of a sea state. Both the regular and the irregular wave group schemes were applied to two different ship models to estimate the probability of exceedance for several roll angle thresholds and comparisons with Monte Carlo simulations of roll motion were presented. The results indicate good coincidence in the tail region where the efficiency of direct simulations is generally low. For intermediate roll angle thresholds the "critical wave groups" method performs better when irregular wave groups are employed due to realistic modelling of wave period successions. However, the probability calculations are sensitive to the degree of variability that is allowed in the wave period groupings. The extent up to which wave group period variations are responsible for resonant phenomena is a topic of future research.

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