Melnikov’s Method Applied to a Multi-DOF Ship Model

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ABSTRACT

In this paper, a coupled roll-sway-heave model derived by Chen *et al* (1999) is studied. In order to address the small damping constraint, the extended Melnikov’s method for slowly varying system is used by assuming the damping term is large. Using the extended Melnikov’s method, the critical wave amplitudes are calculated. A phase space transport method has been applied. The ratios of erosion safe basin areas have been calculated based on the Melnikov’s method and were compared with the results from numerical simulations.

KEYWORDS

Multi-DOF Melnikov; Slowly-varying; Ship; Stability.

INTRODUCTION

Six degree of freedom (DOF) vessel motion problems exhibit numerous complexities, particularly when studied analytically. Most previous work on multi-DOF vessel motions either reduced the problems to lower (one or two) DOF problems or used numerical simulations. In the work of ship motion analysis, compared to 1DOF problems, relatively little work have been done using analytical methods for multi-DOF ship motion problems.

In this paper, the extended Melnikov’s method (Salam, 1987) is applied to a roll-sway-heave coupled ship model derived by Chen *et al.* (1999). By changing the coordinates and applying the singular perturbation technique, Chen showed the model can be simplified to a slowly varying system with three variables, which contain roll displacement and roll velocity as the fast varying variables and a slowly varying variable. This kind of system can be manageable using the Melnikov's method discussed by Wiggins and Holmes (1987, 1988). But similar to the planar Melnikov’s method, the constraint of this method is the small perturbation assumption. In order to address this constraint, the extended Melnikov’s method for slowly varying systems is applied. The extended Melnikov’s method developed in the literature by Salam (1987) has been recently applied to ship motions problems such as capsize (Wu and McCue, 2007, 2008, Wu, 2009) and surf-riding (Wu *et al.* 2010 and Wu 2009). The purpose of this work is to show the possibility of applying the extended Melnikov’s method to multi-DOF ship models.

MATHEMATICAL MODEL

The equations of motion for the coupled roll-sway-heave model in the earth-fixed coordinate system can be expressed in Eq.(1)

$$
\begin{aligned}
m y_c^{''} &= Y \\
m z_c^{''} &= Z \qquad (1) \\
I_{44}\phi^{''} &= K
\end{aligned}
$$

in which, m is the mass of the ship, y_c, z_c and ϕ are sway displacement, heave displacement and roll displacement, respectively. Y, Z and K are generalized forces. The prime denotes the derivative with respect to time t. Chen *et al.* (1999) transformed the model to a wave-fixed coordinate, in which the ship is viewed as a particle riding on the surface of the wave. The sway motion is now parallel to the local wave surface and the heave motion is perpendicular to the local wave surface. The equations of motion now can be expressed as in Eq.(2).

$$\begin{aligned}
&\dot{x}_1 = x_2 \\
&\dot{x}_2 = f_1(x_1, y) + \varepsilon g_a(x_1, x_2, y, z_1, z_2, \tau) \\
&\dot{y} = \varepsilon g_b(x_1, x_2, y, z_1, z_2, \tau) \qquad (2) \\
&\varepsilon \dot{z}_1 = z_2 \\
&\varepsilon \dot{z}_2 = f_2(z_1, z_2) + \varepsilon g_c(x_1, x_2, y, z_1, z_2, \tau)
\end{aligned}$$

where $x_1 = \phi$, $x_2 = \phi'$, $z_1 = z_0 / h$ in which h is the draft of the ship. y is a transformed coordinate which contains sway velocity and other variables. z_0 is small compared to h. $\dot{(.)}$ is the derivative relative to τ, where $\tau = \omega_r t$. ω_r is the natural frequency of roll.

In Eq.(2), the heave motion is considered to exhibit fast dynamics compared to roll and y. Chen *et al.* (1999) used the singular perturbation theory to this system to show that z_1 and z_2 can be solved from the steady state equation and can be substituted into the slow dynamics. The dynamics of the whole system Eq.(2) can be represented by the reduced system Eq.(3).

$$\begin{aligned}
&\dot{x}_1 = x_2 \\
&\dot{x}_2 = f_1(x_1, y) + \varepsilon g_a(x_1, x_2, y, \varepsilon, \tau) \qquad (3) \\
&\dot{y} = \varepsilon g_b(x_1, x_2, y, \varepsilon, \tau)
\end{aligned}$$

Chen *et al.* found that for the reduced system in Eq.(3), roll motions are the fast varying variables while y is the slowly varying variable. Systems like this are called slowly varying systems. When $\varepsilon = 0$, this is simply the planar roll motion with zero forcing and zero damping. When ε is a small positive number, the y motion (which includes sway and other motions) becomes relevant. Because the sway motion is stable, the system will trend towards the invariant manifold of roll dynamics.

THEORETICAL BACKGROUND

Melnikov's Method for Slowly Varying Systems

Melnikov's method is one of few analytical methods that can be used to predict the occurrence of chaotic motions in nonlinear dynamic systems. Melnikov's method has been applied to a number of ship dynamics problems, such as capsize in beam seas (Falzarano, 1990) and surf-riding in following seas (Spyrou, 2006). Most of these are treated as single DOF problems. Melnikov's method for multi-DOF problems has been introduced in several references including the works of Wiggins and Holmes (1987, 1988), who derived the Melnikov's function for slowly varying system in Eq.(3).

When $\varepsilon = 0$, the unperturbed system in Eq.(3) has a planar Hamiltonian, which contains a homoclinic (or heteroclinic) orbit. The Melnikov's function for this system is

$$\begin{aligned}
M(t_0) = &\int_{-\infty}^{+\infty} (\nabla H \bullet \vec{g})(q_0(t), t + t_0)\,dt \\
&- \frac{\partial H}{\partial z}(\gamma(z_0)) \int_{-\infty}^{+\infty} g_b(q_0(t), t + t_0)\,dt
\end{aligned} \qquad (4)$$

$\vec{g} = [0, g_a, g_b]$. H is the Hamiltonian for the unperturbed system. $q_0(t) = (x_1, x_2)$ is the coordinates of the homoclinic orbit for the unperturbed system. And $\bullet$ is the dot product.

Melnikov's Method for Slowly Varying Systems with Large Damping

When the damping term is assumed to be large, it is grouped in the unperturbed system.

Therefore, the unperturbed system is no longer Hamiltonian due to the presence of x_2 in $\tilde{f}_1$.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tilde{f}_1\left(x_1, x_2, y\right) \end{aligned} \qquad (5)$$

The homoclinic orbit, which is essential in the formation of Melnikov's function, disappears as well. Since the homoclinic orbit does not arise naturally, it has to be created artificially. Eq.(3) is then written in the form of Eq.(6).

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \tilde{f}_1\left(x_1, x_2, y\right) + \varepsilon \tilde{g}_a(x_1, x_2, y, \varepsilon, \tau) \\ \dot{y} &= \varepsilon g_b(x_1, x_2, y, \varepsilon, \tau) \end{aligned} \qquad (6)$$

The Melnikov's function for this system is

$$M(t_0) = \int_{-\infty}^{+\infty} \tilde{x}_2 \tilde{g}_a(q_0, t+t_0) \left\{ \exp\left[-\int_0^t a(s)ds \right] \right\} dt + \int_{-\infty}^{+\infty} \tilde{x}_2 \frac{\partial \tilde{f}_1}{\partial y} \int_0^t g_b(s+t_0)ds \left\{ \exp\left[-\int_0^t a(s)ds \right] \right\} dt \qquad (7)$$

in which, $\tilde{q}_0(t) = (\tilde{x}_1, \tilde{x}_2)$ is the coordinates of the new homoclinic orbit of Eq.(5). $a(s)$ is the trace of the Jacobian matrix of Eq.(5). If the unperturbed system in Eq.(5) is Hamiltonian, $a(s) = 0$. Eq.(7) can be reduced to the same form as Eq.(4).

Phase Space Transport

As mentioned earlier, the unperturbed system of Eq.(3) has a planar homoclinic orbit, which contains a stable manifold and a unstable manifold. Wiggins and Holmes (1987) pointed out that when ε is small enough, the perturbed system is ε -close to the local unperturbed manifolds in a small neighborhood. Outside of this region, the perturbed manifold is ε-close to the unperturbed manifold in finite time. The theory of phase space transport for planar systems is applied here to predict the safe region erosion in finite time.

For the unperturbed system, the inside of the homoclinic orbit is the safe region. When the homoclinic orbit is perturbed, the manifolds will intersect resulting in *lobes*. And some initial conditions initially inside the safe region may be outside the safe region for the perturbed system (pseudoseparatrix) after some time. This phenomenon corresponds to a special lobe called *turnstile lobe* (Wiggins, 1992). The area of this lobe is given in Eq.(8) (Wiggins, 1992).

$$\mu(L_0) = \varepsilon \int_0^T M^+(t_0, \phi_0) dt_0 + \mathrm{O}(\varepsilon^2) \qquad (8)$$

in which $M^+(t_0, \phi_0)$ is the positive part of the Melnikov's function, L_0 represents the lobe, t_0 is the parameter in the homoclinic orbit $q_0(t_0)$ denoting different time in the Poincaré map. ϕ_0 is the phase difference with the external forcing. T is the period of the external forcing.

Phase space transport refers to the initial conditions transporting outside the safe region after several periods of external forcing. The amount of the transported phase space can be used to show the rate of safe area erosion. Chen and Shaw (1997) derived the estimate of erosion ratio as shown in Eq.(9).

$$\rho_e = \frac{3\mu(L_0)}{A_s} = \frac{3\varepsilon}{A_s} \int_0^T M^+(t_0, \phi_0) dt_0 + \mathrm{O}(\varepsilon^2) \qquad (9)$$

where A_s is the area of the unperturbed safe region, ρ_e is the ratio erosion area divided by the original safe region, and because ε is a small positive number, $\mathrm{O}(\varepsilon^2)$ term can be ignored. In this work, Eq.(9) is used to show the erosion of safe basin for the capsize problem.

APPLICATIONS

The data from twice capsized fishing boat *Patti-B* are used here for numerical investigation. Chen *et al.* (1999) proposed this model shown in Eq.(10).

$$f_1 = k_{11} - x_1 + k_{12}x_1^2 + k_{13}x_1^3$$
$$g_a = \sigma_{41}\cos x_1 + \sigma_{42}\cos^2 x_1 - \delta_{42}y - \delta_{44}x_2 - \delta_{44q}x_2|x_2| - \lambda f_1(x_1)\cos\Omega\tau + \gamma_{41}\sin\Omega\tau \qquad (10)$$
$$g_b = \sigma_{21}\cos x_1 - \delta_{24}x_2 - \delta_{22}y + \gamma_{21}\sin\Omega\tau$$

f_1 is the restoring moment in the roll motion, which includes the effect of bias. $-\delta_{44}x_2 - \delta_{44q}x_2|x_2|$ is the nonlinear roll damping. Ω is the non-dimensional wave frequency. Other coefficients come from hydrodynamic forces, wind forces and wave forces.

Melnikov's Function

The extended Melnikov's method is applied here by assuming the roll damping terms are large. For the slowly varying system, it is essential to have a homoclinic orbit in order to calculate the Melnikov's function (Wiggins, 1987). If the linear damping term is assumed to be large, the center in the unperturbed system will become a sink, which makes it impossible to have a homoclinic orbit. In this work, the following damping term is assumed for roll

$$B(x_2) = \delta_{44}x_2 + bx_2^2 + cx_2^3 \qquad (11)$$

where b and c are coefficients.

Although it is physically unrealistic to have quadratic damping term in roll, it is used here to show the possibility of using the extended Melnikov's method to multi-DOF problems.

In order to form the homoclinic orbit for the unperturbed system, the quadratic damping term is assumed to be large. The unperturbed system is now

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = k_{11} - x_1 + k_{12}x_1^2 + k_{13}x_1^3 + bx_2^2 - \delta_{42}\bar{y} \qquad (12)$$

where $\bar{y} = \frac{\sigma_{21}}{\delta_{22}}\cos \bar{x}_1$ is the sway variable obtained from averaging. $\bar{x}_1$ is the coordinate of the saddle point, which can be calculated by setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. Eq(12) contains a homoclinic orbit starting from a saddle connecting to itself, as shown in Figure 1. The solid line in the figure is the homoclinic orbit for Eq.(12), while the dashed line is the homoclinic orbit for the unperturbed system in Eq.(3) without the quadratic damping term. These two homoclinic orbits start from the same saddle point, and are close to each other. The Melnikov's function can be calculated using Eq.(7). Numerical integration can be carried out without difficulty.

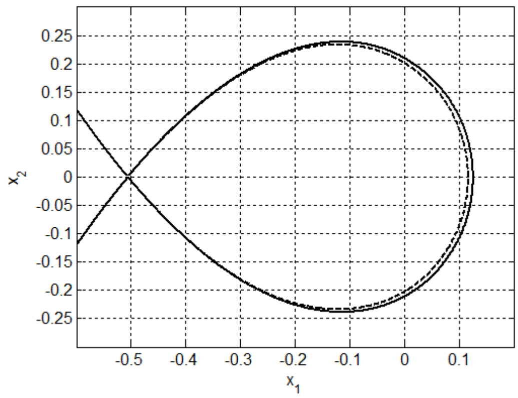


Fig. 1: Homoclinic orbit for the unperturbed system

Numerical Results

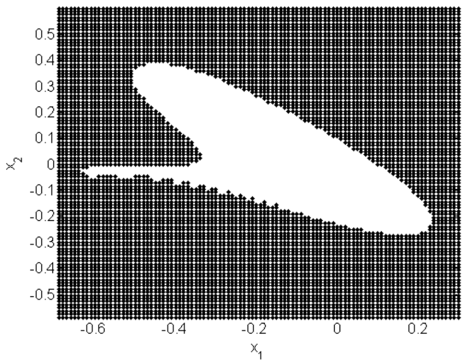
Chen *et al.* (1999) have found the hydrodynamic and hydrostatic coefficients in Eq.(10) for *Patti-B* at wave frequency $\omega_w = 0.6 rad/s$. In this work, the simulation is carried out for the case when the center of gravity has slight bias $y_G = 0.025$. The wind forces are assumed to be zero. The quadratic damping coefficient is set to $b = 0.1$. Melnikov's functions for both the standard and extended methods can be calculated using Eqs (4) and (7), respectively. When $M(t_0) = 0$, this corresponds to the critical wave amplitude a beyond which the chaotic motion and capsize may occur. The critical wave amplitude a has been calculated for both Melnikov's methods listed in Table 1.

As shown in the table, the extended Melnikov's method predicted the critical wave amplitude slightly higher than the standard Melnikov's method for the case studied here.

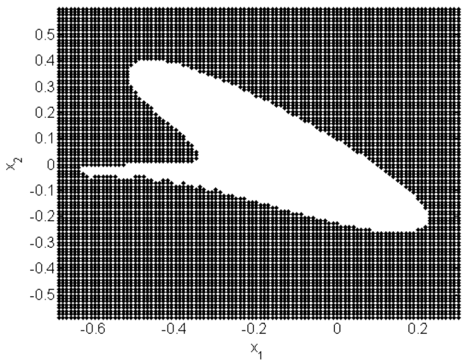
Table 1: **Critical wave amplitude for two Melnikov's methods**

Method	a (m)
Standard Melnikov	0.1792
Extended Melnikov	0.1826

Numerical simulations are carried out to obtain safe basins of the 3DOF system with the damping terms shown Eq.(11) and with original damping term. The safe basins are calculated by integrating a grid of 100×100 points in roll plane with y, z_1 and z_2 initial conditions equal to 1. Every initial condition is integrated until a roll angle is greater than the angle of vanishing stability ($0.5063rad$), thus capsize occurs or through 10 cycles of external forcing, thus deemed safe. Capsize was checked every $dt = 0.01s$. Figure 2(a) is the system with quadratic damping and Figure 2(b) is the original system. In both cases, the wave amplitude $a = 0$.



(a)



(b)

Fig. 2: Safe basins for different models (The white areas are the safe basins, and the dark areas are capsize area.) (a). Safe basin for system with quadratic damping included. (b). Safe basin for original system.

The ratio of erosion area has been calculated using Eq.(9) for both Melnikov's function defined by Eqs.(4) and (7). Numerical simulations are also carried out for the 3DOF system to compare the results. Chen and Shaw (1997) pointed out that in order to implement phase space transport methods, the dynamics should be studied on the invariant manifold where lobes can be defined. Therefore, similar to their work, the initial conditions for the numerical simulations have been chosen as 1720 points on the invariant manifold of roll dynamics, which are obtained by numerically calculated the safe points for the unperturbed system (basically the homoclinic orbit). Two points are picked on every direction of y, z_1 and z_2 . A grid of $(1720\times2\times2\times2)$ points are used as the initial conditions. For the numerical data, the ratio of erosion area is calculated using the points capsized in 10 cycles of external forcing divided by the total number of points.

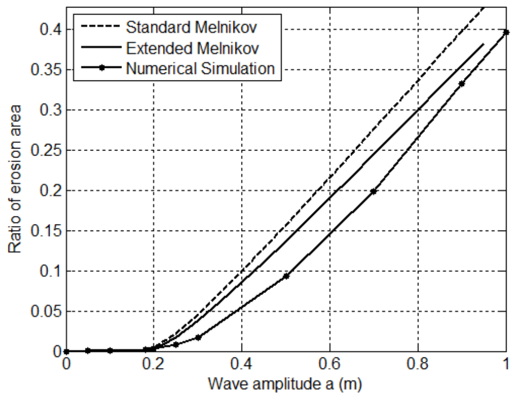


Fig. 3: The ratio of erosion area for different methods.

Figure 3 shows the ratio of erosion areas for different methods. The results from both Melnikov's methods are conservative compared to the numerical simulation results. And the results from the extended Melnikov's method are more accurate than those from the standard Melnikov's method, especially for larger wave amplitudes. Compared to the time consuming 3DOF numerical simulations, the method of phase space transport based on the extended Melnikov's method provides a fast way to estimate ratio of erosion with reasonable accuracy.

CONCLUSIONS REMARKS

In this paper, the extended Melnikov’s method has been used to a roll-sway-heave coupled model which can be reduced to a slowly varying system. In order to obtain the homoclinic orbit, a quadratic damping term is treated as large. Although it is physically unrealistic to have a quadratic term in roll damping, it is used here just to demonstrate the feasibility of the method. Coupled with the method of phase space transport, this results in a fast and effective way to estimate the ratio of erosion with apparently conservative accuracy.

This work is the first step of applying the extended Melnikov’s method to a special form of multi-DOF dynamical systems. It provides the possibility of applying the method to other multi-DOF problems in ship dynamics.

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