

# An Analysis of the Parametric Roll Events Measured Onboard the PCTC AIDA

Jianbo Hua, *SSPA Sweden AB*

Mikael Palmquist, *Seaware AB*

Georg Lindgren, *Lund University*

## ABSTRACT

Between February 1 and 4, 2004, the PCTC AIDA experienced head sea parametric rolling at five different occasions, which was recorded by an onboard operational decision support system.

By applying the common wave spectrum theory together with the ship dynamics theory appropriate for the parametric roll, it could be shown that the parametric roll could not occur to the PCTC AIDA at the onboard evaluated and hind-cast sea conditions expressed in term of a PM-spectrum with 12.6s – 13.5s peak period. However, the heave and pitch motions were shown to be quite regular under the time intervals, during which the recorded successively growing roll motions took place. It can then be deduced that the ship-encountered waves were quite regular during these times. Both wave amplitude and wave period could be estimated from the measured instance heave and pitch motions. By time-domain simulation, it has been shown that the parametric rolls can occur to the present ship in these regular waves and get the magnitudes quite near the recorded ones.

**Keywords:** *Parametric roll, simulation, full-scale measurements, wave groups*

## 1. INTRODUCTION

It is well known that ships based upon the RoRo-concept can be subjected to considerable variation of metacentric height due to the wave profile along the ship. This problem was addressed as early as in 1980s after some capsizing accidents. Huss (1988) conducted a systematic investigation on the influence of hull form of RoRo-ships on the GM-variation in waves. The conclusion was that the hull forms optimized for efficient cargo handling and low resistance are more sensitive to the GM-variation in waves than conventional hull forms.

One consequence is that ships with considerable GM-variation in heading or following

waves are vulnerable for an unstable roll phenomenon called for parametrically excited roll. The magnitude of the GM-variation, ratio of the roll natural frequency to the encounter frequency and the roll damping are the main parameters governing the occurrence of this kind of roll problem.

Under a series of seakeeping model test in MDL, the seakeeping wave basin at SSPA, for a RoRo-ship by Söderberg (1985), the parametrically excited roll motions were measured. The same RoRo-ship was later used by Hua (1992) in a time-domain simulation study based a nonlinear strip numeric model, taking the coupling between the roll, heave and pitch into account, the parametrically excited roll motions were re-constructed in heading and following waves, fairly in agreement with the

model measurements.

In February 2003, the Wallenius PCTC M/V Aida experienced a sudden violent rolling in rough head sea southwest of the Azores. Roll angles as large as 50 degrees were read off the bridge inclinometer. When this incident was post-analysed it was found that the conditions were such that parametric rolling was the most likely cause. Partly due to this incident, M/V Aida was equipped with a measurement system in December 2003 for trial during the winter. Between February 1 and 4, 2004, the PCTC AIDA experienced head sea parametric rolling at five different occasions, although not as critical as in 2003, which was recorded by the onboard system, for details see Palmquist and Nygren (2004).

To the authors' knowledge, this is the first time that parametric rolling is recorded in full scale during normal operation. Actually, the parametric rolling occurred in a rather moderate sea state with a significant wave height well below the threshold wave height for this ship according to IMO MSC/Circ.707. In this paper, an analysis is conducted in order to identify the underlying factors causing the parametric rolling.

The main particulars of AIDA and loading condition at the recording occasion are listed in table 1.

Table 1 Main particulars of AIDA and loading condition at the recording occasion.

Lpp	190 m
B	32.26 m
Draught	9.34 m
Trim	0.68 m
KG	13.6 m
Displacement	34091.4 ton
GM	1.38 m

## 2. DATA FROM ONBOARD RECORDING

Among other functions, the Seaware En-Route Live<sup>®</sup> system conducts 6 d.o.f. ship motion measurements, on site wave spectrum

evaluation based upon the measured motions. In addition to this it provides a recording functionality that continuously records parameters such as ship motions, wind speed and direction from anemometer, ship speed and course from GPS, evaluated seastate and more.

Figure 1 below shows the recorded data during the entire voyage. The first graph shows measured heave, pitch and roll in terms of significant values during 2 minutes blocks of motion samples with a sampling frequency of 10 Hz. As seen, sudden and relatively large rolling occurred 5 times during 2 and 3 Feb 2004. The second and third graphs show mean values of wind speed and relative wind direction during 2 minutes blocks (based on approximately 1 Hz sampling frequency). Relative wave direction is defined so that 0 degrees represents head wind (positive values means wind on starboard side, negative on port side). The fourth (lower) graph displays significant wave height and mean zero-crossing period as estimated by the measurement system. The wind wave direction used for wave estimations is assumed to equal a moving average of the wind direction. **Figure 2** shows the measured time series of heave, pitch and roll at 2004-02-02 14:55:48UTC, and Figure 3 at 2004-02-03 16:06:44UTC.

Hindcast wave spectra at the times of parametric rolling were obtained from the Swedish Meteorological and Hydrological Institute (SMHI) and compared against evaluated wave spectra by the measurement system. The hindcast wave spectra represent analysis fields for the time of concern, computed at European Centre for Medium-Range Weather Forecasts (ECMWF). In Figure 4 and Table 2, hindcast and evaluated wave spectra for the position of M/V Aida at 1200 UTC 2004-02-02 shows fairly good agreement, having almost identical peak periods. Table 3 is a corresponding comparison for 2004-02-03 at the time of large rolling. These also compare fairly well, although the evaluated peak period is approximately 1.5 s larger than for the hindcasts.

As mentioned in the introduction of this pa-

per, M/V Aida experienced far more extreme rolling in head sea the year before, in February 2003. As the loading conditions were quite similar, it is interesting to compare hindcast wave spectra for the two occasions. In Figure 4 it can be seen that the hindcast wave spectra for 2003-02-17, i.e. last years incident, is very similar to the hindcast wave spectra for 2004-02-02. The severe rolling experienced at 2003-02-17 can then be deduced most probably as a parametric roll.

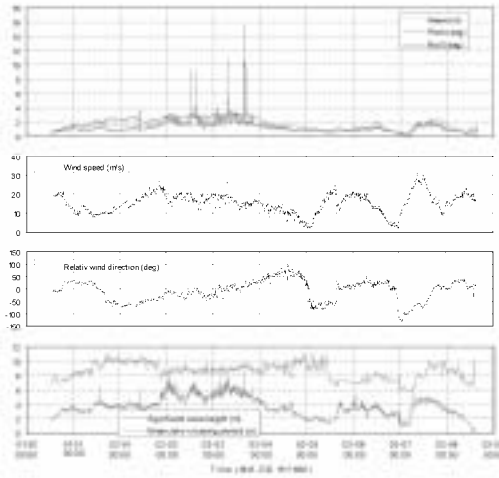


Figure 1 Overview of the voyage in terms of recorded motions, wind and wave conditions as estimated by the measurement system.

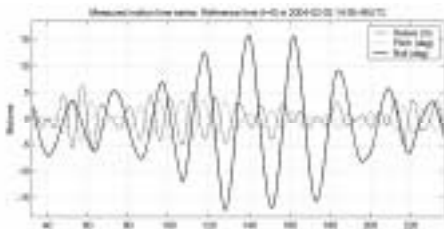


Figure 2 Measured time series of heave, pitch and roll at 2004-02-02 14:55:48UTC. Positive pitch is bow down, positive heave is upwards and positive roll is to starboard side.

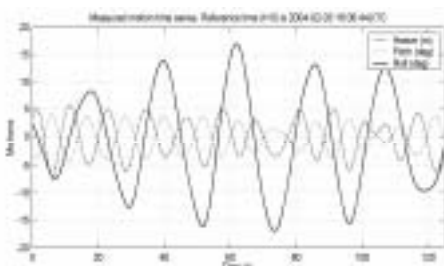


Figure 3 Measured time series of heave, pitch and roll at 2004-02-03 16:06:44UTC. Positive

pitch is bow down, positive heave is upwards and positive roll is to starboard side.

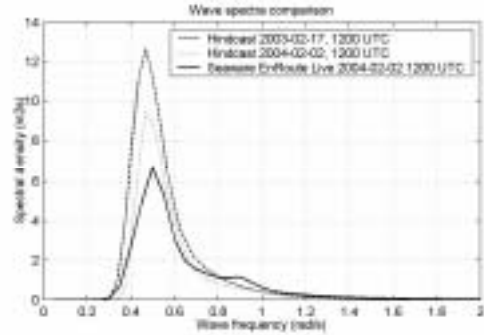


Figure 4 Comparison of wave spectra: Estimated wave spectra 2004-02-02, hindcast wave spectra 2004-02-02 and hindcast wave spectra for incident 2003-02-17. The peak periods were almost identical at the two occasions.

Table 2 Comparison of significant wave height and peak period 2004-02-2 – hindcast vs evaluated

	Sign. wave height, $H_s$ (m)	Peak period, $T_p$ (s)
Hindcast	5.6	13.4
EnRoute Live	~5.2-6.0	~12.7-13.2

Table 3 Comparison of significant wave height and peak period 2004-02-3 hindcast vs evaluated

	Sign. wave height, $H_s$ (m)	Peak period, $T_p$ (s)
Hindcast 12UTC	4.5	12.6
Hindcast 18UTC	4.3	12.6
EnRoute Live 16UTC	~5.1-6.0	~13.5-14.0

### 3. ANALYSIS OF THE ROLL EVENTS

#### 3.1 Characteristics of the GM-Variation

So long as roll amplitude is limited, the parametric roll of a RoRo-ship can be expressed in the following single roll equation

$$\ddot{\phi} + D(\dot{\phi}) \cdot \dot{\phi} + \omega_0^2 \left\{ 1 + \frac{GM(t)}{GM_0} \right\} \cdot \phi = \omega_0^2 \cdot \phi_{bias} \quad (1)$$

In the above,  $GM(t)$  is GM-variation as function of time when a wave passes through the ship. The contributing components to the GM-variation are the hydrostatic effect due to heave and pitch motion, effects of incident wave potential, radiation and diffraction potential. In Hua (1992), a nonlinear strip approach was applied for time domain simulation of parametric roll of a RoRo-ship in heading and following waves taking the coupled roll, heave and pitch into account while the radiation and diffraction effect on the GM-variation were calculated with a rough simplification. The numerical result shows that the radiation and diffraction effect were insignificant and the simulated parametric rolls were in fair agreement with the model measurements. Theoretically, the radiation and diffraction effect on the parametric excitation are in magnitude one order lower compared to the hydrostatic effect and the effect of incident wave potential respectively, and decrease with increasing wavelength. Thus, it is reasonable to take only the hydrostatic effect and the incident wave potential effect into consideration when analyzing the characteristics of the GM-variation of a ship in waves and for making quantitative study of the parametric roll.

The GM-variation of a ship in an irregular wave can be expressed as a Volterra system as the following, according to Hua (1995),

$$\partial GM(t) = \sum_i \partial GM_i(t) \quad (2)$$

where;

$$\begin{aligned} \partial GM_1(t) = & \sum_{n=1}^N a_n \cdot [f_1(\omega_n) \cdot e^{-i(\omega_n t + \beta_n)} + f_1^-(\omega_n) \cdot e^{i(\omega_n t + \beta_n)}] \\ \partial GM_2(t) = & \sum_{m=1}^M \sum_{n=1}^N a_m \cdot a_n \cdot \left[ u_2(\omega_m, \omega_n) \cdot e^{-i[(\omega_m + \omega_n)t + \beta_m + \beta_n]} + \right. \\ & \left. u_2^-(\omega_m, \omega_n) \cdot e^{-i[(\omega_m - \omega_n)t + \beta_m - \beta_n]} \right] \\ & + \sum_{m=1}^M \sum_{n=1}^N a_m \cdot a_n \cdot \left[ v_2(\omega_m, \omega_n) \cdot e^{i[(\omega_m - \omega_n)t + \beta_m - \beta_n]} + \right. \\ & \left. v_2^-(\omega_m, \omega_n) \cdot e^{i[(\omega_m + \omega_n)t + \beta_m + \beta_n]} \right] \end{aligned}$$

and so on for the higher order GM-variation.

In the above,  $a_m$  and  $a_n$  are the amplitudes of the regular wave components in an irregular wave.  $f_1(\omega_n)$  and  $f_1^-(\omega_n)$  are the complex first order transfer function of the GM-variation.  $u_2(\omega_m, \omega_n)$  and  $u_2^-(\omega_m, \omega_n)$  are the complex second order transfer function for the high frequency part, and  $v_2(\omega_m, \omega_n)$  and  $v_2^-(\omega_m, \omega_n)$  for the slow varying part, see Appendix for details.

Generally, the first order GM-variation is a governing factor to the parametric roll of a ship in heading waves. Figure 5 shows the first order transfer function for the GM-variation of the present ship in heading waves. The forward speed is 10 knots. As can be seen, the transfer function gets its maximum at wavelengths of near 0.6 times the ship length. However, the GM-variation becomes much lower than the half of the maximum when the wavelength becomes longer than the ship length.

The peak frequency of the wave spectrum is about 0.5 rad/s at the occasion when the parametric roll events were recorded, according to the hind cast. PM-spectrum is assumed in this study for approximate description of the wave spectrum. The wavelength corresponding to the peak frequency becomes 247.6m and its ratio to the ship length about 1.3. By Figure 5, the first transfer function can be estimated to be somewhat over 0.2 m/m at this frequency.

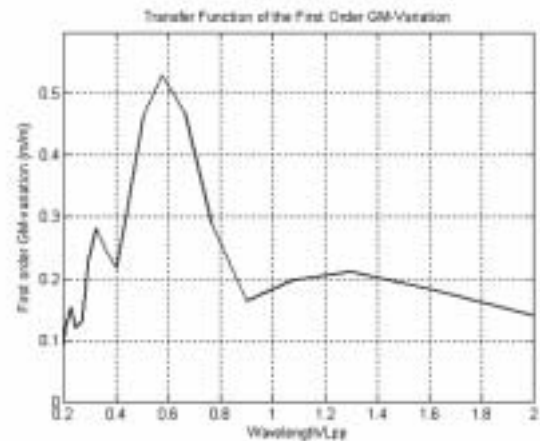


Figure 5 Transfer function of the first order GM-variation.

The spectrum of the GM-variation corresponding to the previous wave condition is shown in Figure 6.. As can be seen, the highest spectrum peak is located at the wave frequency 0.7 rad/s, not the same as the corresponding wave spectrum, and the spectrum shape becomes wider. The major part of the spectrum area is located between 0.6-0.9 rad/s. That means that GM-variation in time domain is very irregular regarding the instance zero-cross period. A typical time series of the GM-variation is demonstrated in Figure 7, which shows that the zero-cross instance period changes from 5s to 8s from one cycle to another and its ratio to the natural roll period becomes between 0.215- 0.348. These ratios are relatively far below 0.5, which is required for parametric roll.

Figure 10 shows that this zero-cross mean period increases with increasing peak period of PM-spectrum. However, the maximal zero-cross mean period is still below 7s for the peak period up to 15s. This means that zero-cross period variation alone cannot explain parametric roll in the present ship in heading waves with peak period up to 15s, assuming a PM-spectrum is relevant for description of the wave condition.

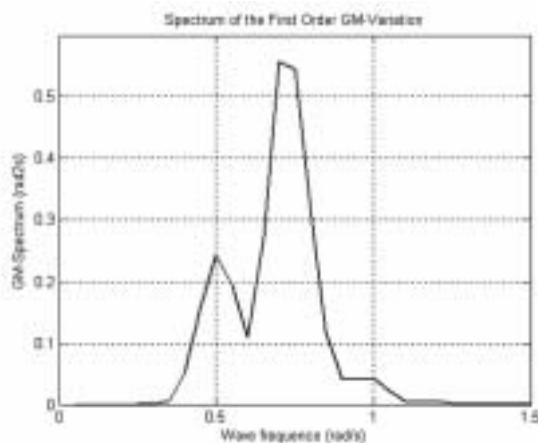


Figure 6 The energy spectrum of the first order GM-variation calculated from the wave energy spectrum (PM-spectrum with  $H_s=5.5$  m and  $T_p=12.6$  s) and the transfer function of the first order GM-variation in Figure 5.

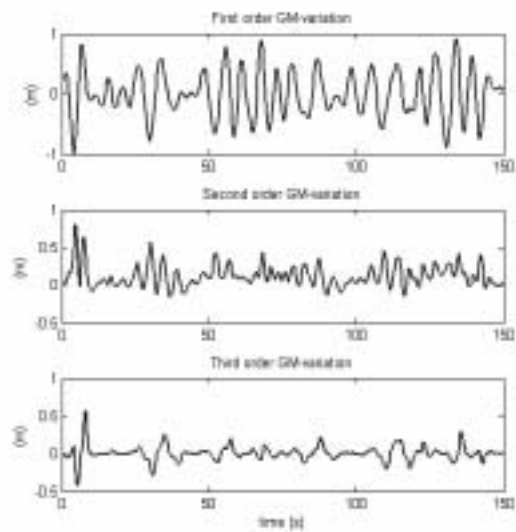


Figure 7 Time series of the first, second and third order GM-variation as function of time.

The zero-cross instance periods shown in Figure 7 are quite irregular and it could be argued that a more regular wave pattern is needed for parametric roll to occur. Such a regular pattern is in fact possible around GM-crests which are considerably higher than average. Figure 8 shows GM-variations with amplitudes higher than 1m. The figure shows 50 s of the variation around the peak instance. Figure 9 shows a histogram of the zero-cross periods close to the high peak, compared to a histogram of all zero-cross periods. The coefficient of variation is 10% with high amplitudes, much smaller than the 33% for all periods, indicating that wave groups become more regular with the presence of one high amplitude cycle. A more detailed study of the length of wave groups for high amplitude cycles is under way, and will be presented elsewhere.

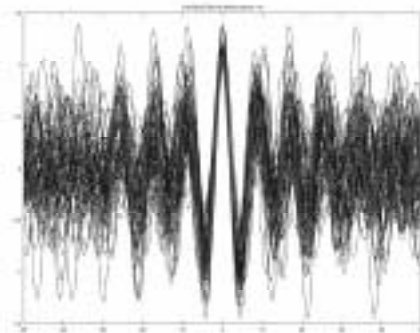


Figure 8 Individual GM-variation above 1m.

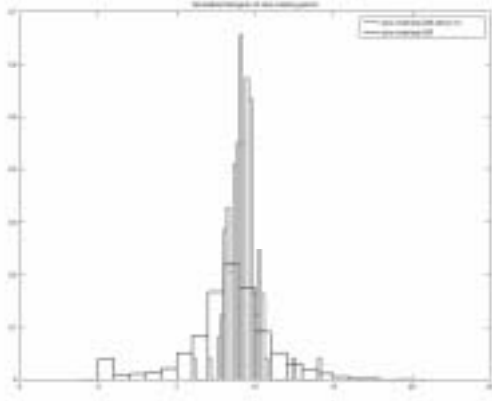


Figure 9 Normalized histogram of the zero-cross periods close to the high peak, compared to a histogram of all zero-cross periods.

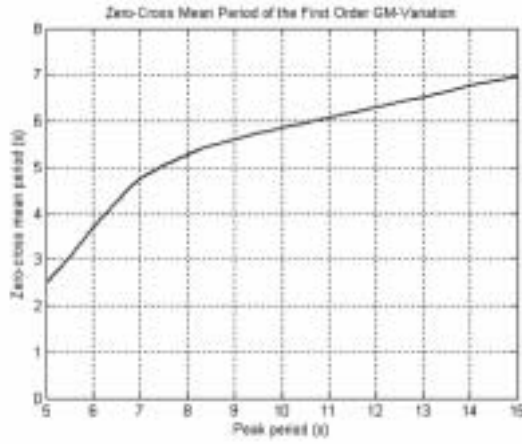


Figure 10 Zero-cross mean period of the first order GM-variation as function of wave peak period of PM-spectrum.

### 3.2 Roll Simulation

Since the measured maximal roll amplitude is below 20 degrees, it is relevant to apply the single roll equation in (1) for time domain simulation for analysis of the measured parametric rolls of the ship in the encountered wave conditions.

$D(\dot{\phi})$  in (1) is the non-linear roll damping coefficient as function of roll velocity, and expressed as the following

$$D(\dot{\phi}) = 2 \cdot \omega_0 \cdot \rho_0 + 2 \cdot \rho_1 \cdot |\dot{\phi}| \quad (3)$$

where  $\rho_0$  is the coefficient for the linear part, and  $\rho_1$  for the quadratic part.  $\omega_0$  the natural roll frequency in calm water,

Here,  $\rho_0=0.03$  and  $\rho_1=0.405$  are used so that the equivalent linear roll damping become 15% at 20 degrees roll amplitude according to the following definition

$$\rho_e(a_\phi) = \rho_0 + \frac{8}{3 \cdot \pi} \rho_1 \cdot a_\phi$$

so that

$$D(\dot{\phi}) = 2 \cdot \omega_0 \cdot \rho_e(a_\phi) \quad (4)$$

where  $a_\phi$  is the expected roll amplitude. This roll damping is fair assumed according to our experience from model measurements for this kind of ship.

Time-domain simulations of the single roll equation in (1) have been conducted, using the time history of the GM-variation in the hind cast wave conditions as input and an initial roll angle of 3 degrees. Actually, no parametric roll could be observed in these simulations. This result could be expected since the zero-cross mean period of the GM-variation is about 6.5s, too low to approach 11.6s, which is required to meet the condition for the occurrence of parametric roll of the present ship. Besides, the instance zero-cross period of the GM-variation changes considerably from one period to another as shown in Figure 7.

However, the heave and pitch motion were quite regular in the time intervals during which the roll motions are successively growing, see Figure 2 and Figure 3. Both heave and pitch motion frequencies are almost twice the natural roll frequency. This can be deduced as that the instance wave encountered by the ship must be nearly regular regarding the wave period. The instance wave amplitude is estimated to be about 4m to 5m, by comparing the measured heave and pitch amplitude with the calculated transfer functions of heave and pitch respectively. The amplitude of the first order GM-

variation becomes then about 0.6m or more and its ratio to the initial GM becomes over 0.4.

Thus regular waves are assumed to cause the GM-variations in equation (1) and time domain simulations of parametric roll have been conducted. Figure 11 shows a time history of a simulated roll motion of the present ship in a regular wave. The wave frequency is 0.44 rad/s and the wave amplitude 4m. By increasing the wave amplitude from 4m to 5m, the highest roll angle becomes some degrees higher after 100s, comparing Figure 12 with Figure 11. Actually, the roll motions in Figure 11 and Figure 12 are very similar to the ones in Figure 2 and Figure 3 at the time intervals during which the parametric roll took place.

The pitch and heave motion are taken into account when calculating the GM-variations. Since the encounter frequency is relatively low, the radiation and diffraction effect on the GM-variation should be small and insignificant regarding the parametric roll. Thus, the roll motions in Figure 11 and Figure 12 from the time domain simulation of the single roll equation in (1) should be considered as good approximations of what occurred to the PCTC AIDA during Feb 2 to Feb 3 2004. Consequently, it can be concluded that close to regular waves might cause the recorded parametric roll events.

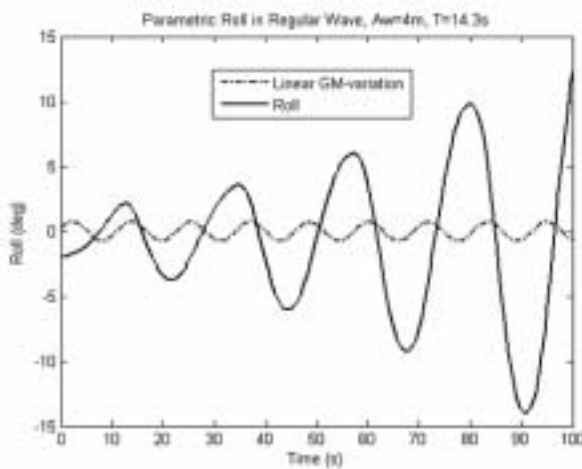


Figure 11 Parametric roll in a regular heading wave. The wave amplitude is 4m.

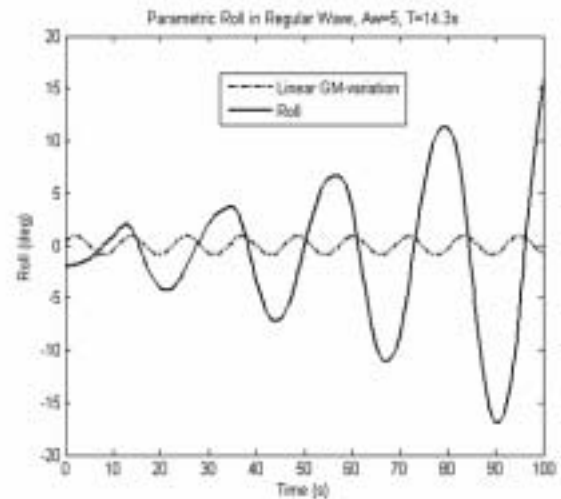


Figure 12 Parametric roll in a regular heading wave. The wave amplitude is 5m.

#### 4. DISCUSSION AND CONCLUSION

By applying the common wave spectrum theory together and the ship dynamics theory appropriate for the parametric roll, it could be shown that the parametric roll could not occur to the PCTC AIDA at the onboard evaluated and hint-cast sea conditions expressed in term of a constant PM-spectrum with 12.6s peak period. That is because the zero-cross mean period of the GM-variation becomes about 6.5s, which is far below the required 11.6 s period half of the natural roll period. Besides, the GM-variation is too irregular in term of instance zero-cross period to be an excitation source required to cause the parametric roll motion of the ship.

However, the heave and pitch motions were shown to be quite regular under the time intervals, during which the recorded successively growing roll motions took place. It can be deduced as that the ship-encountered waves were quite regular during these times. The instance wave amplitude is estimated to be about 4m-5m under the roll events at 2004-02-02 14:55:48UTC and 2004-02-03 16:06:44UTC, and the instance wave period about 13s to 14.3s. By time-domain simulation of the single roll equation in (1), it has been shown that the parametric rolls can occur to the present ship in

these regular waves and get the magnitudes quite near the recorded ones.

The PCTC AIDA experienced similar wave condition in February 2003, and a sudden violent rolling up to 50 degrees was observed. According to the present analysis, it can also be deduced that that event could be caused by several almost regular waves of greater magnitude.

Normally, a wave spectrum provided by weather service is in general an average spectrum over at least 15 minutes up to 2 hours and over a large sea area. As matter of fact, an instance wave spectrum from two minutes wave record can be quite different from the average one. However, knowledge is lacking about the probability of the instance wave spectrum in term of instance peak period, which is required for the risk assessment of the parametric roll such ones as the PCTC AIDA has experienced.

## 5. REFERENCES

Hua, J., 1992" A Study of the Parametrically

Excited Roll Motion of a RoRo-Ship in Following and Heading Waves ", International Shipbuilding Progress, 39, no.420 pp.345-366

Hua, J., 1995, " On the Dynamic Stability Problems of RoRo-Ships , Doctoral thesis, Div. of Naval Architecture, Dept. of Vehicle Engineering, Royal Institute of Technology, S-100 44 Stockholm, Sweden.

Huss, M., 1988, "The Stability of Ships in Waves; a comparative study of modern hull forms with large B/T ratio", TRITA-SKP 1060, The Royal Institute of Technology, Sweden.

Palmquist, M. and Nygren, C. 2004, Recordings of head-sea parametric rolling on a PCTC, SLF 47/INF.5, IMO SUBCOMMITTEE ON STABILITY AND LOAD LINES AND ON FISHING VESSELS.

Söderberg, P. 1985, "Ro-Ro/Containerfartyg och färjor - rullning ", Transportforskningsdelegation, SSPA Report 3534-. In Swedish

## 6. APPENDIX

### 6.1 GM-variation Expressed as a Volterra system

Mathematically, the sectional beam of a ship as a function of draught can be expanded around the mean draught  $T(x)$  as a Taylor Series;

$$B(x, T(x) + z) = B(x, T(x)) + \frac{\partial B}{\partial z} \cdot z + \frac{1}{2!} \frac{\partial^2 B}{\partial z^2} \cdot z^2 + \frac{1}{3!} \frac{\partial^3 B}{\partial z^3} \cdot z^3 + \dots \quad (5)$$

where  $z$  is a variable for the sectional draught change.

As well for the sectional moment with the keel line as reference;

$$M(x, T(x) + z) = M(x, T(x)) + \frac{\partial M}{\partial z} \cdot z + \frac{1}{2!} \frac{\partial^2 M}{\partial z^2} \cdot z^2 + \frac{1}{3!} \frac{\partial^3 M}{\partial z^3} \cdot z^3 + \dots \quad (6)$$

and the sectional area;



---


$$A(x, T(x) + z) = A(x, T(x)) + \frac{\partial A}{\partial z} \cdot z + \frac{1}{2!} \frac{\partial^2 A}{\partial z^2} \cdot z^2 + \frac{1}{3!} \frac{\partial^3 A}{\partial z^3} \cdot z^3 + \dots \quad (7)$$

For the sake of simplicity, we express (5), (6) and (7) as followed;

$$B(x, T(x) + z) = B(x, T(x)) + c_1(x) \cdot z + c_2(x) \cdot z^2 + c_3(x) \cdot z^3 + \dots \quad (8)$$

$$M(x, T(x) + z) = M(x, T(x)) + d_1(x) \cdot z + d_2(x) \cdot z^2 + d_3(x) \cdot z^3 + \dots \quad (9)$$

$$A(x, T(x) + z) = A(x, T(x)) + e_1(x) \cdot z + e_2(x) \cdot z^2 + e_3(x) \cdot z^3 + \dots \quad (10)$$

where;

$$\begin{aligned} c_1(x) &= \frac{\partial B}{\partial z}, & c_2(x) &= \frac{1}{2!} \frac{\partial^2 B}{\partial z^2}, & c_3(x) &= \frac{1}{3!} \frac{\partial^3 B}{\partial z^3} \dots \\ d_1(x) &= \frac{\partial M}{\partial z}, & d_2(x) &= \frac{1}{2!} \frac{\partial^2 M}{\partial z^2}, & d_3(x) &= \frac{1}{3!} \frac{\partial^3 M}{\partial z^3} \dots \\ e_1(x) &= \frac{\partial A}{\partial z}, & e_2(x) &= \frac{1}{2!} \frac{\partial^2 A}{\partial z^2}, & e_3(x) &= \frac{1}{3!} \frac{\partial^3 A}{\partial z^3} \dots \end{aligned}$$

$c_i(x)$ ,  $d_i(x)$  and  $e_i(x)$  can be obtained numerically by using piecewisely polynomial functions fitting the section form, moment and area along the ship.

The initial metacentric height  $GM_0$  at the mean draught in still water is calculated as followed:

$$GM_0 = KB + BM - KG \quad (11)$$

where  $KB = \frac{\int_L M(x, T(x)) \cdot dx}{\Delta}$  and  $BM = \frac{\int_L B(x, T(x))^3 \cdot dx}{12 \cdot \Delta}$ .

The initial GM-variation of a ship in a longitudinal regular or irregular wave can then be determined by integrating the sectional contribution over the ship length;

$$\begin{aligned} \partial GM &= \frac{1}{\Delta} \cdot \int_L \left( \frac{B(x, T(x) + r(x))^3}{12} + M(x, T(x) + r(x)) - A(x, T(x) + r(x)) \cdot KG(x) \right) \cdot dx \\ &\quad - GM_0 \end{aligned} \quad (12)$$

where the sectional mass center above the keel;

$$KG(x) = KG + x \cdot (\eta_5 - \alpha_{trim})$$

$\eta_5$  is pitch motion and  $\alpha_{trim}$  trim. The Smith-effect is neglected in (12). First displacing the variable  $z$  in (8), (9) and (10) with  $r(x, t)$ , the relative motion of the wave surface against ship, and putting (8), (9) and (10) into (12). After having expanded it, we get the following expression;

---


$$\partial GM = \sum_i \partial GM_i \quad (13)$$

where

$$\partial GM_1(t) = \sum_{n=1}^N a_n \cdot \left[ f_1(\omega_n) \cdot e^{-i \cdot (\omega_n \cdot t + \beta_n)} + f_1^-(\omega_n) \cdot e^{i \cdot (\omega_n \cdot t + \beta_n)} \right]$$

and

$$\begin{aligned} \partial GM_2(t) = & \sum_{m=1}^M \sum_{n=1}^N a_m \cdot a_n \cdot \left[ u_2(\omega_m, \omega_n) \cdot e^{-i \cdot [(\omega_m + \omega_n) \cdot t + \beta_m + \beta_n]} + u_2^-(\omega_m, \omega_n) \cdot e^{-i \cdot [(\omega_m - \omega_n) \cdot t + \beta_m - \beta_n]} \right] \\ & \text{etc.} \\ & + \sum_{m=1}^M \sum_{n=1}^N a_m \cdot a_n \cdot \left[ v_2(\omega_m, \omega_n) \cdot e^{i \cdot [(\omega_m - \omega_n) \cdot t + \beta_m - \beta_n]} + v_2^-(\omega_m, \omega_n) \cdot e^{i \cdot [(\omega_m + \omega_n) \cdot t + \beta_m + \beta_n]} \right] \end{aligned}$$

$f_1(\omega_n)$  and  $f_1^-(\omega_n)$  are the first order complex transfer functions for the GM-variation.  $u_2(\omega_m, \omega_n)$ ,  $u_2^-(\omega_m, \omega_n)$ ,  $v_2(\omega_m, \omega_n)$  and  $v_2^-(\omega_m, \omega_n)$  are the second order complex transfer functions. Detailed information about the calculation of the transfer functions can be found in Hua 1995.