On the Effect of Stochastic Variations of Restoring Moment in Long-Crested Irregular Longitudinal Sea

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ABSTRACT

In this paper, the problem of dangerous conditions induced by variations of restoring in waves is addressed, with particular attention to parametric roll. An analytical model for the study of roll motion in irregular longitudinal long crested waves is presented. Stochastic stability boundaries in the first parametric resonance region are provided analytically. Nonlinear/non gaussian roll response above threshold is determined numerically and compared with experiments. A semi-empirical methodology is proposed to extend the classical deterministic static stability assessment to a rational probabilistic framework taking into account hydrostatic restoring variations in waves and different environmental conditions. Examples of application are provided.

Keywords: *irregular sea, parametric roll, nonlinear motions, stochastic stability, pure loss of stability, stability criteria, performance based criteria*

1. INTRODUCTION

Ships' restoring capabilities have, in the modern developments of Naval Architecture, been assessed by analysing the calm water curve of righting moment. This type of ship's characteristic is useful and physical with sufficient approximation when heeling moments are of concern in calm water or when excitation is mainly due to beam sea. It was however clear to several authors in the past (Kuo et al., 1986; Palmquist, 1996; Paulling, 1961) that large variations of the underwater geometry due to ship motions combined with geometrical/hydrodynamic waves' effect, could have detrimental effects on the safety of the ship, leading to large rolling motions and even capsize. Due to the fact that dangerous phenomena in intact ship condition are mainly occurring in wavy sea, it is important to address the case where the restoring moment of the ship changes from an only angle-dependent quantity, to a time/space-and-angle-dependent function. Variations of righting moment are often more evident in regular longitudinal waves and in the case of irregular long crested longitudinal sea. The latter case is a reasonable simplification of the actual short-crested sea surface, that is conservative when only static variations of righting moment are of concern.

This paper deals with the dynamic and static effects of such variations. Dynamic conditions are considered as those situations leading to the build-up of parametrically excited roll motions. Static aspects of the problem are relevant when dealing with the so called pure loss of stability. These two conditions occur, usually, in quite different ranges of speed, especially for those ships suffering of parametric roll in head sea. There is of course a range of frequency where the motion is neither purely quasi-static, nor parametrically excited in any of the instability regions.

To address the basic physics of the phenomenon, an analytical model is proposed, that has already been used in the past for analysing the problem of roll's ergodicity

(Bulian et al., 2006) and to study the susceptibility of a small fishing vessel (Bulian & Francescutto, 2005) to suffer of parametric roll.

Starting from the Grim effective wave concept (Grim, 1961), and by making use of hydrostatic calculations, a simplified analytical 1.5-DOF model is developed that incorporates nonlinear stochastic fluctuations of the curve. From this modelling a series of analytical/numerical considerations are carried out.

The first, important characteristic of this modelling is that it shifts the usual concept of a deterministic curve, to a stochastic model for the righting lever (as was done in the past by Umeda and Yamakoshi (1986) for the case of pure loss of stability). The righting moment then becomes a stochastic process, whose characteristics depend on hull shape and sea spectrum. The classical deterministic curve turns out to be a particular case of this more general concept. By making use of this idea, several classical deterministic approaches (such as usual stability criteria) can be extended in a probabilistic framework to obtain probabilistic engineering measures of ship's restoring capabilities, taking into account the variations induced by waves.

The dynamic model can be employed to study dynamic aspects such as parametric roll in irregular sea. Although an extensive literature is available for the case of regular sea (see, e.g, the references in Bulian (2005)), a more limited number of analyses have been carried out in the case of irregular sea (among others, Belenky et al. 2003; Bulian et al., 2006; Dunwoody, 1989; Palmquist, 1996; France et al., 2003; Hua et al., 1999) especially concerning fundamental aspects of the phenomenon, such as stochastic stability boundaries (Bulian et al., 2004). The availability of this analytical physically sound model allows to address, by using known results (Roberts, 1982), the problem of stochastic stability boundaries, after suitable linearisation procedures: such an analysis allows to give an indication of the critical speeds likely to lead to parametric stochastic instability for a given type of sea spectrum and significant wave height. When conditions are above threshold, numerical simulations can be carried out to analyse the non-gaussian behaviour of the motion (Bulian et al., 2006). A comparison between numerical simulations and experimental results will show, however, that for the ship under analysis, the model is too conservative, and a tuning coefficient must be introduced for the magnitude of the parametric excitation. After tuning the model, very good comparison is obtained with experiments.

The analysis of the modifications that the spectrum of the effective wave amplitude undergoes as the speed is changed, gives useful hints in determining critical conditions in terms of groupiness, by looking at the spectral bandwidth (Bulian et al., 2004). In addition, an analysis of the mean oscillation frequency of the process allows to find a region where a purely static model could be roughly applied (Umeda & Yamakoshi, 1986).

2. A SIMPLIFIED DYNAMIC 1.5-DOF MODEL

2.1 Development of the Model

Ship motions in irregular longitudinal sea have the peculiarity that the roll restoring capability of the ship is often quite strongly explicitly time and angle dependent. Such dependence (parametric excitation) is due to geometrical effects and coupling with longitudinal motions (heave and pitch) in a way that is similar to that known for the regular sea case. In the case of regular sea, parametric roll has been widely investigated in the past experimentally, numerically, and theoretically (see, e.g., references in Bulian (2005 and 2006)). In the particular case of analytical approaches to parametric roll in longitudinal regular sea, basically two types of models can be found, namely 1-DOF and N-DOF models, where usually N=3, i.e., heave, pitch and roll. A single degree of freedom nonlinear model based on the quasi static assumption for heave and pitch and using hydrostatic calculations for the restoring has been developed and investigated in Bulian (2004,2005 and 2006). On the other hand, 3-DOF approaches can be found in Neves & Rodriguez (2004) and Tondl et al. (2000). In the case of 1-DOF models it is impossible to deal correctly with dynamic effects in heave and pitch, whereas such coupling is of course the key feature of 3-DOF models. On the other hand a 1-DOF model is more amenable to direct analytical approaches involving nonlinearities up to sufficiently high order (Bulian, 2004; Umeda et al., 2003; Spyrou, 2000). A fully analytical approach above threshold is more difficult to be applied to multi-DOF systems, and strong nonlinearities can hardly be tackled, but at the same time the threshold level can be predicted by multi-DOF analytical models without extremely cumbersome calculations. However, the increased accuracy of multi-DOF models is obtained at the cost of an increased number of parameters, while a 1-DOF model can be used with very few parameters. Despite the simplicity the 1-DOF modelling, comparisons with experimental results carried out in Bulian (2003) and Bulian et al. (2004), seem to indicate that it could be used as a practical first approach tool. On the basis of the outcomes for the 1-DOF model in regular wave, it has been decided to try to follow a similar way even in the case of irregular waves. Details of the derivation are reported in Bulian (2006). The basic assumptions on which the model is developed are:

- Long crested irregular longitudinal sea;
- Constant ship speed;
- Quasi static assumption for heave and pitch;
- The use of Grim's effective wave to reduce the number of degrees of freedom of the forcing process (waves) from infinite to one;
- Hydrostatic pressure under the Grim effective wave;

On the basis of such assumptions, the models is (Bulian, 2006; Bulian et al., 2006):

$$\ddot{\phi} + d\left(\dot{\phi}\right) + \omega_0^2 \cdot \frac{\overline{GZ}\left(\phi, \eta_{eff}\left(t\right)\right)}{\overline{GM}} = 0 \qquad (1)$$

where $d\left(\dot{\phi}\right)$ is the damping term, and $\eta_{eff}\left(t\right)$ is the amplitude of the Grim effective wave. The spectrum S_η of the Grim effective wave amplitude η_{eff} (a zero mean gaussian process) is determined, in longitudinal sea, from the sea elevation spectrum S_Z at the real frequencies as:

$$S_\eta\left(\omega\right) = 4\left(\frac{Q \cdot \sin Q}{\pi^2 - Q^2}\right)^2 S_Z\left(\omega\right); Q = \frac{\omega^2 \cdot L}{2g} \qquad (2)$$

Doppler effect is applied to S_η to account for the effect of the assumed constant speed of advance. The final fully analytical model is obtained by combining (1) with the following 2D polynomial fitting of the surface $\overline{GZ}\left(\phi, \eta_{eff}\right)$:

$$\begin{aligned} \overline{GZ}\left(\phi, \eta_{eff}\right) &= \sum_{n=0}^{N_G} K_n\left(\eta_{eff}\right) \cdot \phi^n \\ K_n\left(\eta_{eff}\right) &= \sum_{j=0}^{N_K} Q_{jn} \cdot \eta_{eff}^j \end{aligned} \qquad (3)$$

2.2 Parametric Resonance: Stability Threshold

When the dominant frequency of the parametric excitation is close to of twice the roll natural frequency, the upright position can lose stability in a stochastic sense. Closed form results are available in Ibrahim (1985) and Roberts (1982) for the analytical determination of stability boundaries for a parametrically excited 1-DOF system, based on a preliminary application of the stochastic averaging technique. In order to apply such known results, roll motion equation is firstly deterministically linearized with respect to the roll angle in the vicinity of $\phi = 0$, leading to:

$$\ddot{\phi} + 2\mu\dot{\phi} + \omega_0^2 \frac{K_1\left(\eta_{eff}\left(t\right)\right)}{\overline{GM}}\phi = 0 \tag{4}$$

Subsequently the parametric excitation term $K_1\left(\eta_{eff}\right)$ is substituted by its "statistically equivalent linearization" according to Roberts & Spanos (1990):

$$\begin{aligned}
&K_1\left(\eta_{eff}\right) = \sum_{j=0}^{N_K} Q_{j1}\cdot\eta_{eff}^j \rightarrow D_0 + D_1\cdot\eta_{eff} \\
&\underline{\underline{A}}\begin{pmatrix} D_0 \\ D_1 \end{pmatrix} = \underline{\underline{B}}; \quad \underline{\underline{A}} = \begin{pmatrix} 1 & E\left\{\eta_{eff}\right\} \\ E\left\{\eta_{eff}\right\} & E\left\{\eta_{eff}^2\right\} \end{pmatrix} \\
&\underline{\underline{B}} = \begin{pmatrix} \sum_{j=0}^{N_K} Q_{j1}\cdot E\left\{\eta_{eff}^j\right\} \\ \sum_{j=0}^{N_K} Q_{j1}\cdot E\left\{\eta_{eff}^{j+1}\right\} \end{pmatrix} \\
&\frac{E\left\{\eta_{eff}^k\right\}}{\sigma_\eta^k} = \begin{cases} 0 \;\; \text{for } k \text{ odd} \\ \frac{2^{\frac{k}{2}}}{\sqrt{\pi}}\cdot\Gamma\left(\frac{k+1}{2}\right) \;\; \text{for } k \text{ even} \end{cases}
\end{aligned} \tag{5}$$

where $E\{.\}$ is the expected value operator (often called "mean"). The "fully" linearized system is then:

$$\ddot{\phi} + 2\mu\cdot\dot{\phi} + \frac{\omega_0^2\cdot D_0}{\overline{GM}}\left(1 + \frac{D_1}{D_0}\cdot\eta_{eff}\right)\cdot\phi = 0 \tag{6}$$

The coefficients D_0 and D_1 depend on the actual standard deviation σ_η of the effective wave according to (5) and thus on the significant wave height. This meaning that the applied statistical linearization partially accounts for the non gaussian behaviour of the parametric excitation term K_1. If we let now C to be the roll amplitude, the asymptotic stochastic stability limit for $E\left\{C^n\right\}$ can be determined as:

$$\frac{\pi}{8}\left(1 + \frac{n}{2}\right)\frac{\omega_0^2\cdot D_1^2}{\overline{GM}\cdot D_0}S_\eta\left(2\omega_0\sqrt{\frac{D_0}{\overline{GM}}}\right) < \mu \tag{7}$$

In particular, a sufficiently practical condition could be assumed (arbitrarily) to be that associated to $n = 1$, i.e., the condition for asymptotic stochastic stability of the mean of the roll envelope. When such condition is satisfied, the expected value of the roll envelope for the system (6) strictly decays to zero as the time is increased.

It is important to stress that, in the determination of the linear system (6), with the consequent application of the result (7), use has been made of the statistical linearization technique that is, basically, an "ensemble" (thus instantaneous) linearization technique not taking into account time domain effects. For this reason frequency mixing induced by the nonlinear transformation $K_1\left(\eta_{eff}\left(t\right)\right)$ cannot be properly accounted for. If the majority of the energy spectrum of η_{eff} is concentrated close to twice the roll natural frequency, and if nonlinear effects in the transformation K_1 are not extremely significant, then the proposed approach can be considered suitable. On the other hand, when the energy of the parametric forcing η_{eff} is concentrated far from the sub-harmonic resonance condition and nonlinearities in K_1 are significant, then the direct Fourier transform of the autocorrelation function for the process $K_1\left(t\right)$ should be determined and used to obtain the magnitude of the parametric excitation in (7).

2.3 Response above Threshold

When the upright position loses stability, the ship starts rolling, and nonlinear effects become fundamental in bounding the motion's statistics. It can indeed be proved (Ibrahim, 1985) that, after performing stochastic averaging, the linearized model (6) cannot show a steady state probability density function for the roll motion, similarly to what happens in the case of regular deterministic excitation (Bulian, 2006; Francescutto, 2002). In order to obtain a steady state solution (i.e. steady state statistics for the roll motion, being it a stochastic process), nonlinear effects must be introduced. Linearization techniques (typical of the beam sea case) cannot be used for damping, due to the completely different role played by linear and nonlinear component. Boundedness of roll's statistics is obtained for $t \to \infty$ thanks to:

- Nonlinear damping (increased rate of energy dissipation at large rolling amplitudes/velocities);
- Nonlinear restoring (detuning effect);

In the case of regular sea, the deterministic averaging technique has been used to obtain an approximate nonlinear roll response curve in frequency domain (Bulian, 2004) in the region of the first parametric resonance. A similar approach could in principle be used for the stochastic case, by means of the stochastic averaging technique (Roberts & Spanos, 1984). In its original form, however, such technique is not able to account for the effects on roll amplitude statistics of constant terms in nonlinear restoring (Roberts & Spanos, 1984), and the only analytically detectable bounding effects are those given by the nonlinear damping. Although generalisations of the previous technique exist that are able to detect detuning effects (Ibrahim, 1985; Roberts & Spanos, 1984; To, 1998), the application of these latter techniques leads often to cumbersome calculations to arrive to an, in any case, approximate solution. This makes the Monte Carlo approach, at least for the time being, more appealing. In this work, roll response curves, when reported, have been determined by means of Monte Carlo approach.

2.4 Some Comments on other Regions Far from Subharmonic Resonance

In the previous sections we concentrated on the dangerous condition of 2:1 subharmonic resonance. The model (1) could, however, be used in different regions of encounter frequency. In particular it is interesting to discuss the case of so called "pure loss of stability". Such region is intended here as that zone of parameters (sea spectrum characteristics and ship speeds) where dynamic effects can be considered to be small, and the rolling motion has thus long characteristic period. Umeda and Yamakoshi (1986) used in the past the Grim simplification to address this phenomenon, that is typical in following sea in suitable speed ranges. In order to determine such speed range, it could be useful to analyse the characteristic frequencies (mean and zero-crossing) of the parametric excitation:

$$\begin{aligned} &\text{mean: } \overline{\omega}_{e\eta} = m_1/m_0 \\ &\text{zero-crossing: } \omega_{e\eta,z} = \sqrt{m_2/m_0} \\ &m_n = \int_0^\infty \left| \omega + \frac{\omega^2 U}{g} \right|^n S_\eta(\omega)\, d\omega \end{aligned} \tag{8}$$

The ship speed U is considered as positive in head sea. Regions of minimum values for $\overline{\omega}_{e\eta}$ and $\omega_{e\eta,z}$ are those where the characteristic period of the variations of restoring is longest, and where, then, the quasi static approach is, hopefully, more likely to be in agreement with the reality.

Another important parameter in the qualitative analysis of dangerous conditions is the spectral bandwidth of the forcing process, i.e. a measure of the energy concentration (that can be proved to be related to the difference between the mean and zero crossing frequency of the forcing process). In particular, it was shown in Bulian et al. (2004); Bulian (2006); Panjaitan (1998) that conditions quite far from exact 2:1 resonance, but associated to small values of spectral bandwidth (i.e. narrow band forcing spectra) can be more dangerous than conditions where the excitation frequency is closer to the 2:1 resonance, but where the bandwidth of the parametric excitation is large, and thus grouping is less marked.

2.5 Application

Some examples of application of the presented analytical model are reported in this section. Calculations are performed on a 132m long RoRo having the main particulars reported in Table 1.

Table 1: Main particulars of ship TR2.

Ship TR2			
Length:	132.22m	Volume:	7714m^3
Breadth:	19m	GM :	0.865m
Draught:	5.875m	Roll period:	15.9s

Speed dependent roll damping coefficients have been obtained from experiments while righting arm calculations have been carried out using a free trim approach. Experimental data have been obtained from tests carried out in the INSEAN model basin (Bulian, 2006; Bulian et al., 2004 and 2006). Figure 1 compares experimental and numerical roll standard deviation (i.e. roll rms) for a Bretschneider sea spectrum having modal wave frequency corresponding to a wave length equal to the ship length.

It can be seen that the predictions with the proposed model (indicated as “Not tuned”) overestimate the experimental roll motion standard deviation (used here only as an indicator of roll motion amplitude, being the motion strongly non gaussian). Such overestimation can be due to an underestimation of the damping, or an overestimation of the parametric excitation. Since damping coefficients have been obtained from experiments, it is assumed here that the source of error is the modelled magnitude of the parametric excitation. For this reason, an empirical tuning coefficient k_C is introduced, and the effective wave amplitude process $\eta_{eff}(t)$ is substituted by the tuned process $k_C \cdot \eta_{eff}(t)$ (this actually corresponds, for the Bretschneider sea spectrum, to a substitution, in the calculations, of the physical significant wave height $H_{1/3}$ with a “correct significant wave height” equal to $k_C \cdot H_{1/3}$).

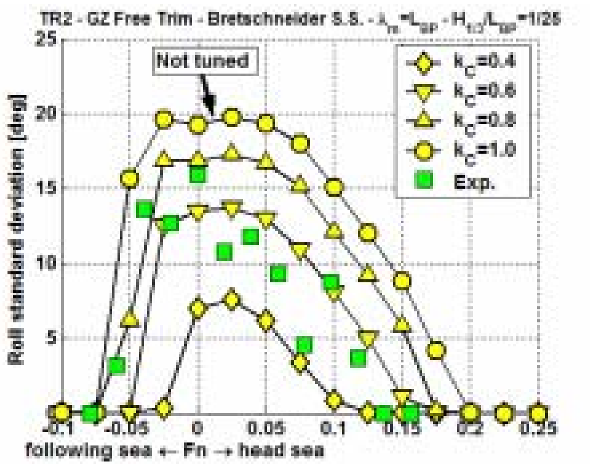

Figure 1: Comparison between predictions and experiments. Roll standard deviation.

Due to the original overestimation of roll, we tried different values of $k_C \le 1$, in order to reduce the magnitude of the parametric excitation. It can be seen that the use of a suitable value for the tuning coefficients leads to a good correlation between prediction and experiments, and that a tuning coefficient of about 0.6 gives, in this case, a good mean correlation, apart from a relative shift between the experimental and the numerical response curves in terms of Froude number. Such good correlation is much more evident when we compare the numerically and experimentally determined cumulative distribution functions of the roll envelope, as in Figure 2. The tuning coefficient $k_C = 0.75$ in the figure has been selected by matching the numerical and the experimental roll standard deviation. It is to be said that the data used for the zero speed analysis and those used for the tests with forward speed come from different test campaigns (Bulian et al., 2004 and 2006) on the same model: differences in the wave generation, or other sources of errors could have influenced the outcomes, leading to experimental results at zero speed not completely in line with those obtained with non-zero speed of advance.

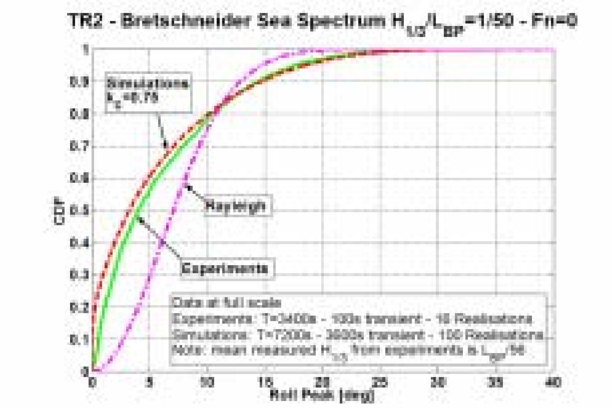

Figure 2: Comparison between tuned predictions and experiments. CDF of roll envelope.

In Figure 2 a Rayleigh distribution is superimposed in order to show that the usual narrow band gaussian approach is not suitable for parametric roll analysis (Belenky et al., 2003; Bulian et al, 2006; Hashimoto et al., 2005). On the other hand, the numerically predicted cumulative distribution function of roll envelope after tuning, very well agrees with the experimental results. It seems then that, although the magnitude of the parametric excitation as obtained by using the quasi static approach is too large for the ship under analysis, the general form of the mathematical model, in terms of nonlinear damping and restoring functions, is sufficiently suitable to reproduce the non gaussian roll motion statistics when the parametric excitation has the correct magnitude. However, such a large reduction in the magnitude of parametric excitation in irregular waves was not expected at the beginning of this study, having in mind the quite good average agreement obtained by a similar model in regular waves (Bulian, 2006). For this reason further research is worth on this point, in order to understand whether an “average” value for such tuning factor could be found, such to provide a first approach, fast tool for parametric roll analysis in irregular waves. It is however to be pointed out that, for the ship under analysis, the provided model is conservative.

The developed analytical methodology allows to obtain in a quite fast way, indications regarding the limiting threshold significant wave height as function of ship speed and modal frequency of the wave spectrum. In Figure 3 such limiting surface is reported without any tuning on the parametric excitation. The presence of two local minima is related to the humps and hollows in the spectrum of the effective wave amplitude (see Figure 4). Such limiting surface can easily be linked with classical wave scatter diagrams and the (joint) probability density function of the ship speed (accounting for voluntary and/or involuntary speed reduction) in order to define an index of susceptibility I_S to parametric roll phenomenon, defined as the probability of sailing in a condition above the limiting surface (Bulian, 2006).

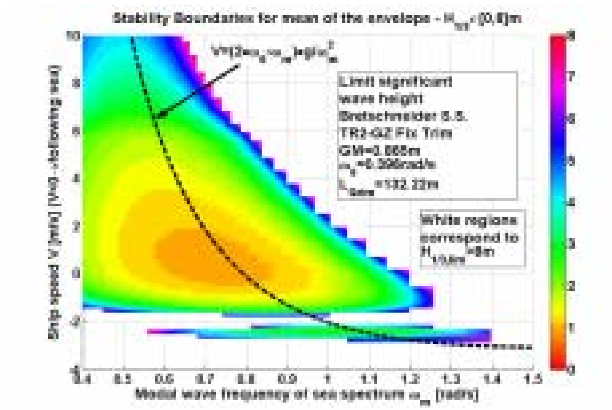

Figure 3: Estimated limit significant wave height for the inception of parametric roll.

As discussed in the previous section, the characteristic frequencies of the parametric excitation process, together with the analysis of the spectral bandwidth, could give useful information on the region where a quasi-static approach could me more successful. Figure 4 shows, on the left, a comparison between the wave spectrum and the effective wave spectrum when the modal wave length is equal to the ship length (the ship in this case is TR2), together with an analysis of the mean and zero crossing frequencies (upper right plot), and of the spectral bandwidth of the effective wave process (lower right plot) as a function of Froude number. A definite minimum can be seen regarding frequencies around Fn=0.35 in following sea, while the minimum of the spectral bandwidth occurs close to Fn=0.18 in following sea. It is important to note that the region of minima for the effective wave frequencies is close to the region of maximum for the spectral bandwidth: it could then be guessed that the most dangerous conditions, with respect to the pure loss of stability, could occur between Fn=0.18 and Fn=0.35 in following sea with sufficiently large significant wave height, due to the occurrence of sufficiently strong grouping together with long characteristic periods for the excitation. In case of different ratios between ship length and modal wave length, results could differ. It is interesting to note that, for the ship under analysis, the second parametric resonance region occurs close to the region of maximum grouping, and this could endanger the ship. Although for the ship under analysis Froude numbers of 0.3 and above are unrealistic, the concept can be used for any ship, including high-speed vessels or small ships.

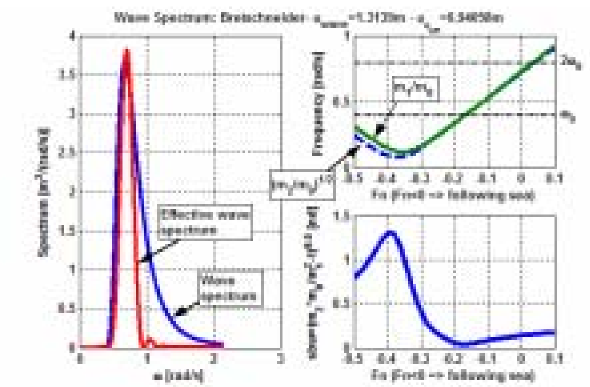

Figure 4: Analysis of characteristic frequencies and spectral bandwidth for the parametric excitation process.

3. A SEMI-EMPIRICAL PROBABILISTIC APPROACH TO STATIC STABILITY

The previous sections have dealt with a "first principles" dynamic analysis of roll motion in irregular longitudinal waves. A direct physical approach, even if approximate, is for sure the most suitable, long lasting, easily upgradeable and elegant way of providing tools for the future generation of stability criteria. It is however important to have in mind that present stability criteria (IMO, 2002), and thus present design practice, are mainly based on static considerations for the righting arm curve in calm water dating back to the Rahola's work (Rahola, 1939) or to the original developments of the Weather Criterion (IMO, 2005). At the same time there is considerable pressure towards the development of new criteria able to reduce the risk of large rolling angles that some ship typologies are presently facing when at sea, and Classification Societies have started facing the parametric roll matter by issuing some Guidelines (ABS, 2004): a global picture of the results and effects of their implementation is, however, not yet available to the public. In the meantime some evidence has been reported (Hinrichs & Krueger, 2004; Krueger, 2005) that statically determined stability variations in waves, could be roughly related to the probability of experiencing large rolling angles, or even capsize: the basic idea being that large restoring variations in waves together with small calm water righting arm, could identify dangerous designs.

An extension of the classical deterministic approach to static stability in order to rationally account for hydrostatic stability variations in irregular waves, in a sound probabilistic framework is thus presented. This allows, in principle, to address different environmental conditions in both long and short term. The idea is based on the nonlinear model (3) for $\overline{GZ}$ as a function of the heeling angle and the effective wave. More details are available in Bulian (2006).

3.1 $\overline{GZ}$ as a Random Quantity

Given a particular sea state, the effective wave spectrum can be obtained according to (2), and it is thus possible to determine the variance σ_η^2 of the effective wave by integration. The effective wave process has a zero mean gaussian probability density function, i.e. $\eta_{eff} \in N\left(0,\sigma_\eta^2\right)$. We consider only ensemble averages, and we are not interested, in this section, in dynamic effects. The instantaneous probability density function of η_{eff} provides, thus, a full characterisation of the input random variable. The knowledge of the relationship $\overline{GZ}\left(\phi,\eta_{eff}\right)$ in (3) allows to derive the metacentric height and the area under the righting arm as functions of the effective wave amplitude:

$$
\begin{aligned}
&\overline{GM}\left(\eta_{eff}\right) = \left.\frac{\partial \overline{GZ}\left(\phi,\eta_{eff}\right)}{\partial \phi}\right|_{\phi=0} = \\
&= K_1\left(\eta_{eff}\right) \\
&b\left(\phi,\eta_{eff}\right) = \int_0^\phi \overline{GZ}\left(\xi,\eta_{eff}\right) d\xi = \\
&= \sum_{n=0}^{N_G} \frac{K_n\left(\eta_{eff}\right)}{n+1} \cdot \phi^{n+1}
\end{aligned}
\tag{9}
$$

Since η_{eff} is a random variable (actually it is a stochastic process, but, as said, we are here interested only in the ensemble domain), all the mentioned quantities, i.e., $\overline{GZ}$, $\overline{GM}$ and b are to be considered as (correlated) random variables having non gaussian distributions. Thanks to their known polynomial relation with the effective wave amplitude, their probability density functions can be obtained numerically without necessarily using Monte Carlo simulations. However, even using a Monte Carlo approach is not very time consuming, and huge samples can fast be generated to obtain sufficiently reliable estimations.

3.2 The Definition of Compliance Level

We assume to have an arbitrary criterion $C\left\{\overline{GZ}\right\}$, i.e., a "rule" able to "judge" the geometrical $\overline{GZ}$ curve properties. Such criterion is assumed to provide values in the interval $[0,1]$, where 0 means not compliance, whereas 1 mean full compliance. The criterion $C\{.\}$ could be binary (i.e. $C\{.\} \to \{0,1\}$, pass/fail type) or fuzzy (i.e. $C\{.\} \to [0,1]$, with different "levels" of compliance). Present IMO criteria (IMO, 2002) based on the analysis of $\overline{GZ}$ are an example of binary criterion $C\{.\}$, because they only provide a binary outcome: pass/fail. Often the criterion $C\{.\}$ is based on the numerical values of the quantities $\overline{GZ}$, $\overline{GM}$ and b. Given $C\{.\}$ we can formally define the following function, that we call the "compliance level I_C":

$$
\begin{aligned}
&I_C\left(\eta_{eff}\right) = C\left\{\overline{GZ}\left(\phi,\eta_{eff}\right)\right\} \\
&I_C : \mathbb{R} \to [0,1]
\end{aligned}
\tag{10}
$$

Due to the random nature of η_{eff}, the compliance level is intended here as a random variable, and, in principle, its cumulative distribution function cdf_C can be found as:

$$
\begin{aligned}
&cdf_C\left(I_C\right) = \int_{\Omega(I_C)} pdf_\eta\left(\eta_{eff}\right) d\eta_{eff} \\
&\Omega(x) = \left\{\eta_{eff} : I_C\left(\eta_{eff}\right) \le x\right\}
\end{aligned}
\tag{11}
$$

Where pdf_η is the gaussian probability density function of the effective wave. Although cdf_C provides full information about the random nature of I_C, from a practical point of view it could be useful to rely, in performing comparisons, on a single quantity, such as the mean value of the compliance level, i.e.:

$$
\begin{aligned}
&\mu_C = E\left\{I_C\right\} = \int_0^1 I_C \cdot pdf_C\left(I_C\right) dI_C = \\
&= \int_{-\infty}^{\infty} I_C\left(\eta_{eff}\right) \cdot pdf_\eta\left(\eta_{eff}\right) d\eta_{eff}
\end{aligned}
\tag{12}
$$

It is known that a wave crest amidship, for conventional ship types, can lead to a significant reduction of restoring. It is then not unlikely for a binary criterion $C\{.\}$ to be fulfilled only up to a certain effective wave amplitude η_{crit}, i.e.,

$$\begin{cases} I_C\left(\eta_{eff}\right)=1 \text{ for } \eta_{\mathrm{eff}} \le \eta_{crit} \\ I_C\left(\eta_{eff}\right)=0 \text{ for } \eta_{\mathrm{eff}} > \eta_{crit} \end{cases} \tag{13}$$

In this particular case (binary criterion with presence of critical effective wave amplitude) the following relations hold:

$$\mu_{\mathrm{C}} = \mathrm{Prob}\{I_C = 1\} = \\ = cdf_\eta\left(\eta_{crit}\right) = \Phi\left(\frac{\eta_{crit}}{\sigma_\eta}\right) \tag{14}$$

where Φ is the cumulative distribution function of the standardized gaussian $N(0,1)$. It is to be stressed that all the previous relations refer to a short term approach, i.e., to a particular sea state. Given a two-parameters discrete wave scatter diagram with probability mass function $P_{SEA}\left(H_{1/3},T_m\right)$, the long term expected value of the compliance level is obtained through the following weighted sum:

$$\mu_C = \sum_{T_m}\sum_{H_{1/3}} \mu_C\left(H_{1/3},T_m\right)\cdot P_{SEA}\left(H_{1/3},T_m\right) \tag{15}$$

Although the previous relation has been written for the long term expected value of the compliance level for a two-parameters wave spectrum, it can be generalised to the other quantities referred in this paragraph, and to a larger number of sea state parameters in continuous or discrete form (Bulian, 2006).

In the case of a ship marginally complying in calm water with a binary criterion $C\{.\}$, under the assumptions of existence of η_{crit}, we always obtain $\mu_C = 0.5$ both in long and short term calculations, because of the fact that η_{eff} has a symmetric probability density function with zero mean. In this particular case, then, $\mu_C = 0.5$ allows to obtain the calm water marginal condition.

According to the considerations reported in (Hinrichs & Krueger, 2004; Krueger, 2005) it is (very) likely that $\left(\mu_C\right)_1 > \left(\mu_C\right)_2$ implies $\{\text{ship(or condition) "1"}\}$ to be safer than $\{\text{ship(or condition) "2"}\}$ with respect to phenomena related to variations of stability in waves.

3.3 Direct Measures of Variation

In the previous paragraph, the magnitude of the hydrostatic variations of restoring in waves has basically been measured by means of an arbitrary criterion $C\{.\}$. Although in any rule, by definition, there is a need to create a “criterion” (there cannot be any rule, i.e. a decision process, if a “criterion” is not specified), we could assume to apply this criterion to short or long term quantities directly related to (almost) physical variations of restoring in waves. In particular, it is important to note that absolute variations of restoring are often meaningless if they are not related to the corresponding mean or, as an approximation, to calm water values. In the case of short term analysis, an example of relevant measure of variation μ_M arising from the considerations reported in (Hinrichs & Krueger, 2004; Krueger, 2005) could be the coefficient of variation of the area under $\overline{GZ}$, defined as follows:

$$\mu_{\mathrm{M}} = \sqrt{\frac{\mathrm{Var}\left\{\mathrm{b}\left(\phi,\eta_{\mathrm{eff}}\right)\right\}}{\mathrm{E}^2\left\{\mathrm{b}\left(\phi,\eta_{\mathrm{eff}}\right)\right\}}} \tag{16}$$

If we assume that, with sufficient practical accuracy, in the range of expected effective wave amplitudes the relation between b and η_{eff} is linear, then

$$\mathrm{b}(\phi,\eta_{\mathrm{eff}}) \approx P_0(\phi) + P_1(\phi)\cdot\eta_{eff} \Rightarrow$$

$$\Rightarrow \mu_M = \left|\frac{P_1(\phi)}{P_0(\phi)}\right|\sigma_\eta \tag{17}$$

where it is important to note that, neglecting effects on trim induced by vertical shifting of the centre of gravity, $P_1(\phi)$ only depends on the hull form, whereas $P_0(\phi)$ depends on $\overline{KG}$: $P_0(\phi)$ is then under the control of the designer even if the hull form cannot be modified.

When long term nondimensional measures of variation are introduced, it is better to rely on the squared value of μ_M, in such a way to exploit the long term standard deviation of the effective wave amplitude. In linear approximation, indeed, we can obtain (Bulian, 2006):

$$\mu_{\mathrm{M,long\text{-}term}} = \sqrt{E\{\mu_M^2\}} = \left|\frac{P_1(\phi)}{P_0(\phi)}\right|\sigma_{\eta,long-term} \tag{18}$$

where $\sigma_{\eta,long-term}$ is the standard deviation of η_{eff} determined by using the long term probability density function of η_{eff}.

In the general case of any quantity q defined by means of a nonlinear transformation $F(\eta_{eff})$ as in (9), the long term n-th statistical moment can be obtained starting from the short term probability density function of the effective wave amplitude $pdf_\eta(\eta_{eff}|\underline{p})$, and by using the joint probability of the sea state parameters vector $\underline{p}$ (e.g. $\underline{p}=(H_{1/3},T_m)$):

$$\begin{aligned} E\{q^n|\underline{p}\} &= \int_{-\infty}^{+\infty} F^n(\eta_{eff})\, pdf_\eta(\eta_{eff}|\underline{p})\, d\eta_{eff} \\ E\{q^n\} &= \int_{p_1}\ldots\int_{p_n} E\{q^n|\underline{p}\}\, pdf_{SEA}(\underline{p})\, dp_n\ldots dp_1 \end{aligned} \tag{19}$$

Results in (17) and (18) are special cases of the previous relations.

The knowledge of a limited number of statistical moments in not sufficient, however, in general, to fully characterise the probability density function of a random variable, even if some asymptotic analytical approximations can be used (with caution), as the Gram-Charlier or Edgeworth expansions (Ibrahim, 1985; Ochi, 1990). It could then be useful to rely, instead, on intervals based on percentile levels, rather than on some statistical moment, in order to provide uniformity even for different types of probability density function for the quantity of interest.

3.4 Application

As an example of application, in this section the estimated long term of the compliance level is determined for three ships in three different environmental conditions. The first ship is TR2, while the second ship is a 52m long RoRo named ITACA having the main particulars reported in Table 2.

Table 2: Main particulars of ship ITACA.

Ship ITACA			
Length:	52.55m	Draught:	2.1m
Breadth:	10m	Volume:	569m^3

The third ship is a scaled GEOSIM of ITACA, and it is called ITACA$_{2.4}$, where the 2.4 scale factor is used to obtain comparable dimensions between TR2 and ITACA$_{2.4}$. The used criterion is the set of statistical criteria provided by present Intact Stability Code (IMO, 2002), but the requirement for the position of the maximum of the $\overline{GZ}$ curve has been neglected (Bulian, 2006). The situation satisfies with sufficient accuracy the case of binary criterion with presence of critical effective wave amplitude, as discussed above, and the dependence of the critical effective wave amplitude on the metacentric height for the three ships is reported in Figure 5. The condition $\eta_{crit}=0$ for each ship identifies the usual marginal $\overline{GM}$ in calm water. When $\overline{GM}$ is reduced below the marginal value in calm water, the critical effective wave amplitude becomes negative: this is due to the fact that the static stability is, usually, increased in a wave trough ($\eta_{crit} < 0$). When $\overline{GM}$ is below the marginal level, the selected criterion can only be satisfied when additional stability is provided by a wave trough amidship.

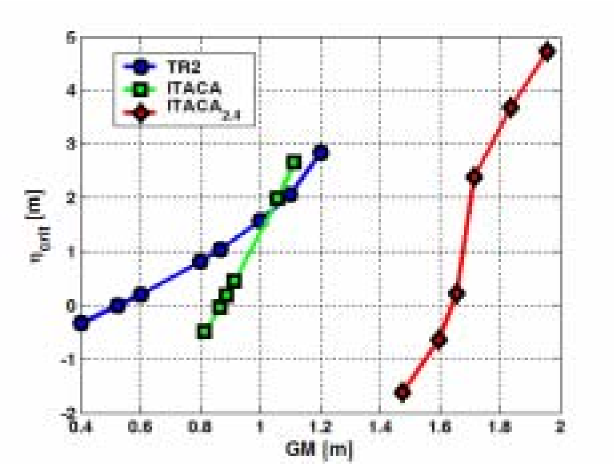

Figure 5: Dependence of critical effective wave amplitude η_{crit} on $\overline{GM}$ according to the selected criterion.

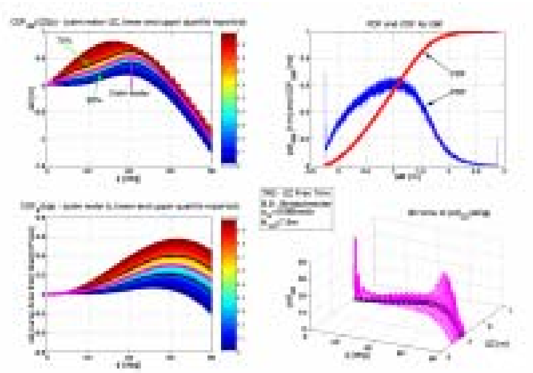

Figure 6: Example of short term representation of static stability in the proposed probabilistic approach. Ship TR2.

The used environmental conditions are obtained from the wave scatter diagrams for the North Atlantic region as recommended by IACS (2001) and for the East Mediterranean Sea (Area 27) as obtained from Hogben et al. (1986). The third considered environmental condition is the North Atlantic as above, with a maximum allowed operational significant wave height equal to 5.5m.

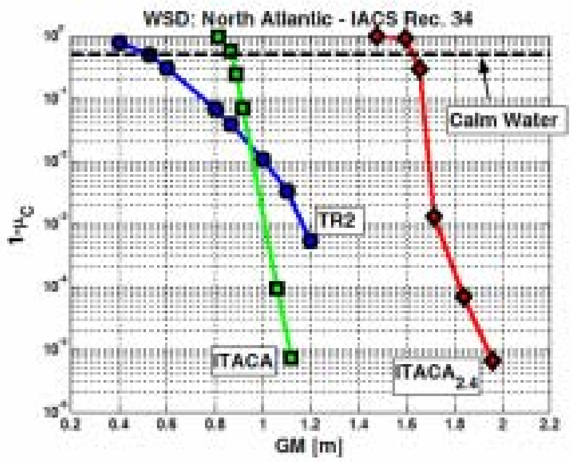

Figure 7: Mean long term compliance level in North Atlantic.

Figure 6 shows how the random quantities related to static stability can be visualised in a global way. $\overline{GZ}$, $\overline{GM}$ and b in this case have been determined by Monte Carlo simulations, but an almost analytical approach can be used as well. The upper right plot shows the probability density function and the corresponding cumulative distribution of the metacentric height, highlighting the non gaussian shape. On the upper left plot the cumulative distribution of $\overline{GZ}$ is reported: it is important to note that such plot is meaningful only when sections are considered at fixed values of the heeling angle (i.e. $cdf\left(\overline{GZ}\middle|\phi\right)$), due to the geometrical correlation between $\overline{GZ}$ at different heeling angles. Similarly, in the lower left plot the cumulative distribution of the area under the $\overline{GZ}$ curve (dynamic stability) is reported, for which the same comments hold. The final lower right plot shows the probability density function of $\overline{GZ}$ at different heeling angles. Calm water curves in the left plots are reported as magenta thick lines, together with the thick curves in black connecting the upper and lower interquartile points. A comparison between the obtained long term value of $1-\mu_C$ in the case of North Atlantic for the three ships is reported in Figure

7. If we select an arbitrary value of (required) $\mu_C \geq 0.5$, the corresponding increase in metacentric height with respect to the usual marginal calm water value can be obtained. Such increase is different for the three ships for any $\mu_C > 0.5$, due to the different hull form geometry and/or main dimensions.

The effect of different environmental conditions is assessed in Figure 8 for the ship TR2. Also in this case, given a particular value of $\mu_C \geq 0.5$, it is possible to obtain the corresponding metacentric height, and the increase in the metacentric height with respect to the marginal calm water value depends on both μ_C and the selected long term environmental condition.

As a final example, the linearized coefficient of variation of the area under $\overline{GZ}$ up to 40deg according to (18) is reported in Figure 9. In this case there is no marginal calm water value, since we are not basing the calculation on any arbitrary *a-priori* criterion. However, provided a limiting (arbitrary, but basically tuned on the present fleet) value for the coefficient of variation, the corresponding limiting metacentric height can be found for different ships and different environmental conditions. It is interesting to note that for the case of the two largest ships under analysis, TR2 and $ITACA_{2.4}$, the change of long term operational area from North Atlantic to East Mediterranean Sea has a large effect on the estimated long term coefficient of variation of $b(40\deg)$. On the other hand the effect is much more limited in the case of the original small RoRo ITACA. This is due to the combined effect of the reduction in the mean significant wave height from North Atlantic to East Mediterranean Sea, that would reduce the coefficient of variation of $b(40\deg)$, however, at the same time, the dominant wave length shortens from North Atlantic to East Mediterranean, leading to a sort of “static resonance” for ITACA, i.e., an increased matching between the effective wave transfer function (that only depends on the ship length) and the dominant sea spectra. For simlar reasons, in the case of TR2 and $ITACA_{2.4}$, that have a length that is more than twice the length of ITACA, the modifications that the Wave Scatter Diagram undergoes between North Atlantic and East Mediterranean Sea lead to a global reduction in the coefficient of variations of $b(40\deg)$.

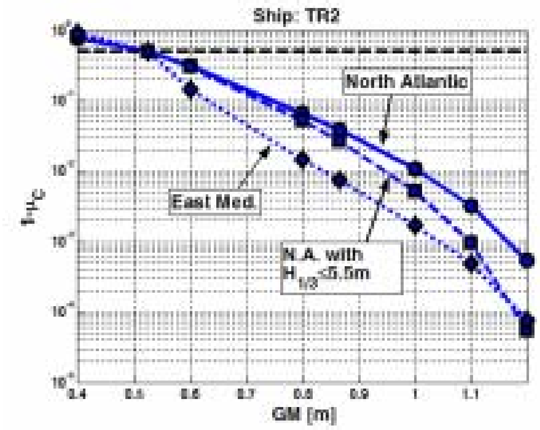

Figure 8: Effect of different assumed environmental conditions on the mean long term compliance level. Ship TR2.

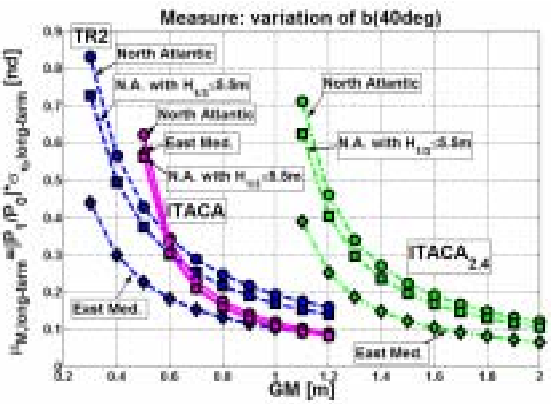

Figure 9: Effect of different assumed environmental conditions on the mean long term compliance level. Ship TR2.

4. FINAL REMARKS

The problem of the effects of variations of restoring moment in longitudinal long crested irregular waves has been dealt with both dynamically and statically by using a simplified nonlinear analytical model for roll motion exploiting the Grim effective wave concept.

In particular, subharmonic parametric roll resonance has been analysed, and a fully analytical approach has been proposed for the determination of the stochastic stability threshold from the original model. Roll motion above threshold, on the other hand, has been analysed by means of Monte Carlo simulations. From the comparison carried out with a series of available experimental data, we observed that the analytical model overestimates roll motion for the ship under analysis. However, after the introduction of a tuning factor on the basic parametric excitation (i.e. basically on the spectrum of the effective wave amplitude), good agreement has been achieved between the experimental data and the numerical predictions. In particular, the non-Rayleigh behaviour of the roll motion envelope seems to be quite well reproduced. The actual origin of the necessity of introducing the tuning factor still remain to be investigated. For the case of regions far from the subharmonic resonance, some considerations have been reported starting from the analysis of the characteristic frequencies and of the spectral bandwidth of the parametric excitation: dangerous conditions related to the “pure loss of stability” are to be searched bearing in mind the grouping effect (roughly measured by the spectral bandwidth) induced by the Doppler effect.

The introduction, by means of the Grim effective wave, of non deterministic righting arm variations has allowed to propose a methodology for extending the classical deterministic concept of static stability in calm water, in a rational probabilistic framework where both the susceptibility of the ship to suffer hydrostatic restoring variations in waves and the sea state characteristics (short and long term) are taken into account. Present deterministic criteria can be then applied in this extended framework, and in addition, rational measures of restoring variations that are independent from any arbitrary a-priori criterion are introduced in a general way, with some particular examples. The developed ideas allows to introduce a ranking among different ships and/or loading/environmental conditions, taking approximately into account the danger inherent in the tendency for the ship to suffer large restoring variations in waves. The usual deterministic approach is a sub-case of the more general approach proposed here. Despite its semi-empirical nature (that needs of course some tuning), the proposed method could be a suitable interim procedure for highlighting ships likely to be endangered by too large hydrostatic restoring variations, waiting for the, hopefully fast, introduction of real first principles performance based approaches.

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